STA3000 Sufficiency and Unbiased Estimation

This is treated in TPE at length, and also in CH §8.4. If $T = t(Y)$ is unbiased for θ, and $S = s(Y)$ is sufficient for θ (in the model $f(y;\theta)$, etc.), then $W = E(T \mid S)$ is unbiased for θ and has smaller variance than T.

To see this,

$$\begin{aligned}\theta = E(T;\theta) &= E\{E(T \mid S);\theta\} = E(W;\theta)\\ \mathrm{var}(T;\theta) &= \mathrm{var}\{E(T \mid S);\theta\} + E\{\mathrm{var}(T \mid S);\theta\} = \mathrm{var}(W;\theta) + E\{\mathrm{var}(T \mid S);\theta\},\end{aligned}$$

where by the sufficiency of S, neither $E(T \mid S)$ nor $\mathrm{var}(T \mid S)$ depend on θ. So W is unbiased, with smaller variance unless $\mathrm{var}(T \mid S) = 0$.

This minimum variance unbiased estimator is also unique, under the assumption that S is a complete sufficient statistic. A sufficient statistic is complete if $E\{(h(S);\theta\} = 0$ for every value θ implies $h(S) = 0$. (A sufficient statistic is boundedly complete if the same holds for bounded functions $h(\cdot)$).This is mainly/only used in establishing uniqueness of optimal tests and estimators based on sufficient statistics; it ensures that if we have two unbiased estimators of θ we end up at the same conditional estimator.

The Lehmann-Scheffé theorem says that if a sufficient statistic is boundedly complete it is minimal sufficient.

In i.i.d. sampling from an exponential family, the minimal sufficient statistic $T = t(Y)$ is complete if and only if the dimension of T is the same as the dimension of θ.