An elegant design due to Hotelling

This is taken from Chernoff.[1]. We have eight objects, whose weights we wish to estimate with as much precision as possible, using a pan balance, which measures the difference between the weight in two pans, with some measurement error, which may be assumed to be random and independent across different weighings.

The design used is the following: eight different weighings are made, with the following objects in the left and right pans, respectively:

left	right
1 2 3 4 5 6 7 8	
1 2 3 8	4 5 6 7
1 4 5 8	2 3 6 7
1 6 7 8	2 3 4 5
2 4 6 8	1 3 5 7
2 5 7 8	1 3 4 6
3 4 7 8	1 2 5 6
3 5 6 8	1 2 4 7

If we denote the true weights by, say $\theta_1, \ldots, \theta_8$, and the responses from the eight weighings by $Y_1, \ldots, Y_8$, then, for example, we can model Y_1 as

$$Y_1 = \theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 + \theta_6 + \theta_7 + \theta_8 + \epsilon_1,$$

where ϵ_1 is a random variable of zero mean and unknown variance σ^2. There is a similar model for each of the Y_is. It is not hard to verify that an unbiased estimate of θ_1, say, (which is also the least squares estimate) is given by

$$\hat{\theta}_1 = (Y_1 + Y_2 + Y_3 + Y_4 - Y_5 - Y_6 - Y_7 - Y_8)/8$$

and that $\text{var}(\hat{\theta}_1) = \sigma^2/8$. Similar unbiased estimates for each θ_i are available, each with variance $\sigma^2/8$, and all uncorrelated.

Thus with just 8 weighings we are able to estimate the weight of 8 objects, with a precision eight times that which we would have if we measured the weight of the objects one at a time.[2]

[1]Chernoff, H. (1972) *Sequential analysis and optimal design*. SIAM Monograph
[2]This design, in technical terms, is a 1/8 fraction of a 2^6 factorial.