Accurate approximation for inference on vector parameters

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Approximate exponential families

- model $f(y;\theta)$, $\theta \in \mathbb{R}^d$
- data $y = (y_1, \ldots, y_n)$ independent observations
- log-likelihood function $\ell(\theta; y) = \log f(y;\theta)$
- linear exponential family:

$$f(y;\theta) = \exp\{\varphi(\theta)' s(y) - c(\theta) - d(y)\}$$

- canonical parameter obtained as

$$\frac{\partial \ell(\theta; y)}{\partial s(y)} = \varphi(\theta)$$

- up to affine transformations
- Example: $N(\mu, \sigma^2)$:

$$\ell(\theta; y) = \frac{\mu}{\sigma^2} \sum y_i - \frac{1}{2\sigma^2} \sum y_i^2 - \frac{n\mu^2}{2\sigma^2} - n \log \sigma$$

Tangent exponential model

- in general, can find an approximate exponential model:

$$f_{TEM}(s;\theta)ds \quad = \quad \exp\{\varphi(\theta)'s + \ell(\theta)\}h(s)ds \qquad (1)$$

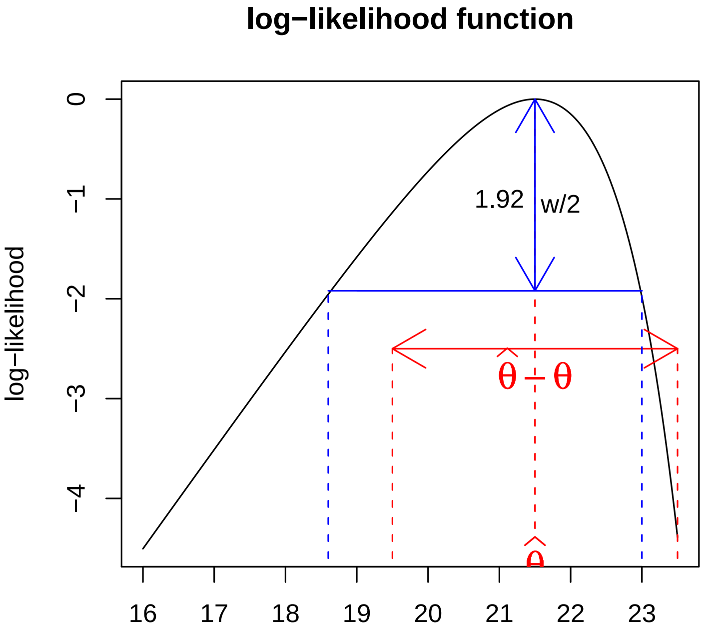
- s is a score variable on $\mathbb{R}^d$: $\quad s(y) = -\ell_{\varphi}(\hat{\theta}^0; y)$
- $\ell(\theta) = \ell(\theta; y^0)$ is the observed log-likelihood function
- $\varphi(\theta) = \varphi(\theta; y^0)$ is the canonical parameter $\in \mathbb{R}^d$
  to be determined
- has the same observed log likelihood function
  as the original model
- has same first derivative on the sample space, at y^0, as the original model
- (1) approximates $f(y \mid a; \theta)$ to $O(n^{-1})$

Likelihood Inference

- assume $\theta = (\psi, \lambda)$, ψ scalar parameter of interest
- likelihood ratio statistic $w(\psi) = 2\{\ell(\hat{\psi}, \hat{\lambda}) - \ell(\psi, \hat{\lambda}_\psi)\}$ approximately χ^2_1
- or a directional departure $r(\psi) = \pm\sqrt{w(\psi)}$
- approximately $N(0, 1)$ $O(n^{-1/2})$
- can do much better:

$$r^*(\psi) = r(\psi) + \frac{1}{r(\psi)} \log\{\frac{q(\psi)}{r(\psi)}\}$$

- $r^* \sim N(0, 1)$ with relative error $O(n^{-3/2})$ under $f(y; \theta)$
- r and q functions only of $\{\ell(\theta; y^0), \varphi(\theta, y^0)\}$
- and their derivatives; also need $\hat{\theta}$ and $\hat{\theta}_\psi$, full and constrained mle, as with w

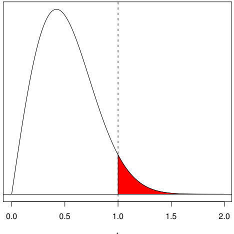


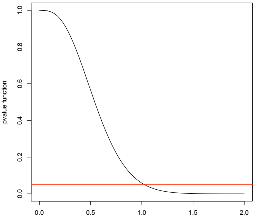
log−likelihood function
log−likelihood
0
−1
−2
−3
−4
16
17
18
19
20
21
22
23
1.92
w/2
$\hat{\theta} - \theta$
$\hat{\theta}$

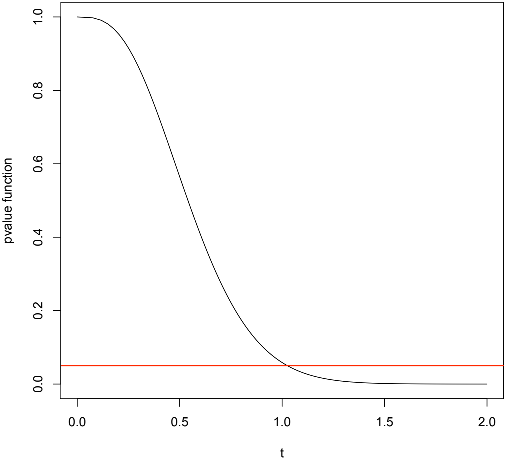
Inference with TEM

- $f_{TEM}(s;\theta) = \exp\{\varphi(\theta)'s + \ell(\theta)\}h(s)$
- $\varphi(\theta) = \varphi(\theta; y^0)$, $\qquad \ell(\theta) = \ell(\theta; y^0)$

- why y^0?
- *p*-value: probability of data as or more extreme than that observed
- can be plotted as a function of the parameter
- provides tests of particular values, and confidence bounds or intervals





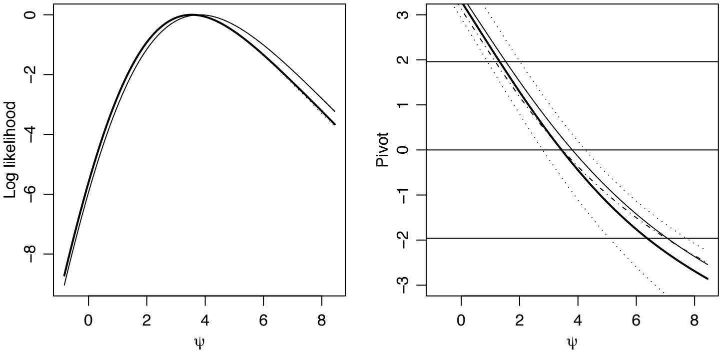


pvalue function
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.5
1.0
1.5
2.0
t

Example: 2×2 table

	M	S
M	1	18
F	5	2

$\psi =$ log-odds ratio



BDR, 2007, Fig.3.4

Details

- $\{\ell(\theta), \varphi(\theta)\} \longrightarrow$ TEM $\longrightarrow$ *p*-value
- using $r^* = r^*(\psi) = r + \dfrac{1}{r}\log(\dfrac{q}{r}) \mathrel{\dot\sim} N(0,1)$
- $r(\psi) = \pm\surd[2\{\ell(\hat\theta) - \ell(\hat\theta_\psi)\}]$ likelihood root
- $q(\psi) = \dfrac{|\varphi(\hat\theta) - \varphi(\hat\theta_\psi) \quad \varphi_\lambda(\hat\theta_\psi)|}{|\varphi_\theta(\hat\theta)|} \dfrac{|j(\hat\theta)|^{1/2}}{|j_{\lambda\lambda}(\hat\theta_\psi)|^{1/2}}$
- observed information $j(\theta) = -\partial^2\ell(\theta)/\partial\theta\partial\theta'$
- nuisance parameter integrated out via Laplace

Canonical parameter $\varphi(\theta)$

- if $f(y;\theta)$ is an exponential family, φ is sitting in the model
- if not
- if y is continuous, define

$$V = \left.\frac{dy}{d\theta}\right|_{y=y^0,\theta=\hat{\theta}^0} \qquad y = (y_1, \ldots, y_n)$$

- ??
- $z_i = z_i(y_i;\theta)$ with a fixed distribution, e.g. $(y_i - \mu)/\sigma$
- $V = -\left(\frac{\partial z}{\partial y}\right)^{-1} \left.\frac{\partial z}{\partial \theta}\right|_{y=y^0,\theta=\hat{\theta}^0} \qquad n \times p$
- $$\varphi(\theta) = \varphi(\theta; y^0) = \left.\frac{\partial \ell(\theta; y)}{\partial V}\right|_{y=y^0} = \sum_{i=1}^{n} \frac{\partial \ell(\theta; y^0)}{\partial y_i} V_i$$

Example: regression

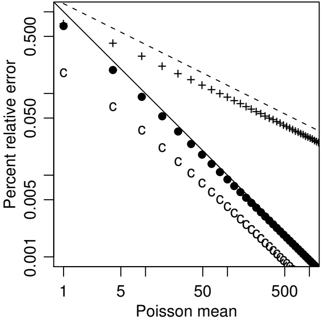
- Model: $y_i = x_i'\beta + \sigma\epsilon_i$
- Canonical parameter: $\varphi(\theta) = \sum_{i=1}^{n} \ell_{;y_i}(\theta; y^0) V_i$
- $V_i = [x_i' \quad (y_i^0 - x_i'\hat{\beta})/\hat{\sigma}]$
- $\varphi(\theta; y) = \sum_{i=1}^{n} \frac{1}{\sigma} g'(\frac{y_i^0 - x_i'\beta}{\sigma})[x_i' \quad \hat{\epsilon}_i]$

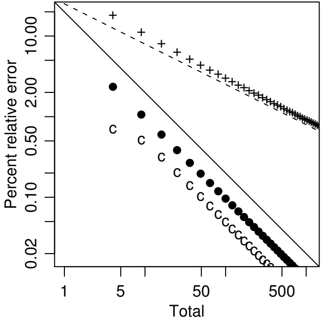
	Normal		t_4 errors	
	Est (SE)	z	Est (SE)	z
Constant	−13.26 (3.140)	−4.22	−11.86 (3.70)	−3.21
`date`	0.212 (0.043)	4.91	0.196 (0.049)	4.02
`log(cap)`	0.723 (0.119)	6.09	0.682 (0.129)	5.31
`NE`	0.249 (0.074)	3.36	0.239 (0.080)	2.97
`CT`	0.140 (0.060)	2.32	0.143 (0.063)	2.26
`log(N)`	−0.088 (0.042)	−2.11	−0.072 (0.048)	−1.51
`PT`	−0.226 (0.114)	−1.99	−0.265 (0.110)	−2.42

... canonical parameter φ

- a sample space derivative of log-likelihood $\ell_{;V}(\theta; y^0)$
- if the sample space is discrete
- $y \longrightarrow s$ score variable
- $\dfrac{dy}{d\theta} \longrightarrow \dfrac{dE(s;\theta)}{d\theta}$ DFR, 2006
- $$s_i = s_i(y_i) = \left.\frac{\partial \ell(\theta; y_i)}{\partial \theta}\right|_{\theta=\hat{\theta}^0}$$
- $$V_i = \left.\frac{\partial}{\partial \theta} E(s_i;\theta)\right|_{\theta=\hat{\theta}_0}$$
- $$\varphi(\theta) = \sum_{i=1}^{n} \left.\frac{\partial \ell(\theta; y^0)}{\partial s_i}\right|_{\theta=\hat{\theta}^0} V_i$$

relative error $O(n^{-1})$





DFR, 2006

Example: Poisson counts

Table 1. Lung cancer deaths in British male physicians (Frome, 1983). The table gives man-years at risk/number of cases of lung cancer, T/y, cross-classified by years of smoking, taken to be age minus 20 years, and number of cigarettes smoked per day.

Years of smoking t	Daily cigarette consumption x						
	Nonsmokers	1–9	10–14	15–19	20–24	25–34	35+
15–19	10366/1	3121	3577	4317	5683	3042	670
20–24	8162	2937	3286/1	4214	6385/1	4050/1	1166
25–29	5969	2288	2546/1	3185	5483/1	4290/4	1482
30–34	4496	2015	2219/2	2560/4	4687/6	4268/9	1580/4
35–39	3512	1648/1	1826	1893	3646/5	3529/9	1336/6
40–44	2201	1310/2	1386/1	1334/2	2411/12	2424/11	924/10
45–49	1421	927	988/2	849/2	1567/9	1409/10	556/7
50–54	1121	710/3	684/4	470/2	857/7	663/5	255/4
55–59	826/2	606	449/3	280/5	416/7	284/3	104/1

$$E_\theta(Y) = T\lambda(x,t) = exp(\theta_1)t^{\theta_2}\{1 + \exp(\theta_3)x^{\theta_4}\}$$

T yrs. at risk   x # cigarettes   t Years smoking θ_4 parameter of interest

... Poisson regression

- $E_\theta(Y) = T\lambda(x,t) = exp(\theta_1)t^{\theta_2}\{1 + \exp(\theta_3)x^{\theta_4}\}$
- linear increase in death rate with 'dose' $\longrightarrow H_0 : \theta_4 = 1$

signed root of log-likelihood ratio statistic	$r = 1.506$	$p = 0.066$
higher order approximation	$r^* = 1.491$	$p = 0.068$

Vector parameter of interest

- $\theta = (\psi, \lambda), \quad \psi \in \mathbb{R}^{d_0} \quad H_0 : \psi = \psi_0$
- usual:

$$W(\psi_0) = 2\{\ell(\hat{\psi}, \hat{\lambda}) - \ell(\psi_0, \hat{\lambda}_{\psi_0})\} \dot{\sim} \chi^2_{d_0}$$

- proposed: profile sample space $\mathcal{S}_\psi$,
- all sample points that give the same estimate for the nuisance parameter $\hat{\lambda}_\psi$
-

$$\mathcal{S}_\psi = \{s : \hat{\varphi}_\psi = \hat{\varphi}^0_\psi\} = \{s : \ell_\lambda(\hat{\theta}^0_\psi; s) = 0\}$$

  a surface of dimension d_0, passing through data point y^0
- Directed departure on $\mathcal{S}_\psi$
- observed value $s^0 = 0$
- expected value s_H under H_0 $\qquad s_H = -\ell_\varphi(\hat{\theta}_{\psi_0})$
- Distribution of magnitude of $|s - s_H|$
- given the direction $(s - s_H)/|s - s_H|$
- Skovgaard, 1988; Fraser & Massam, 1988; Cheah, Fraser, Reid, 1994

Directional p-value

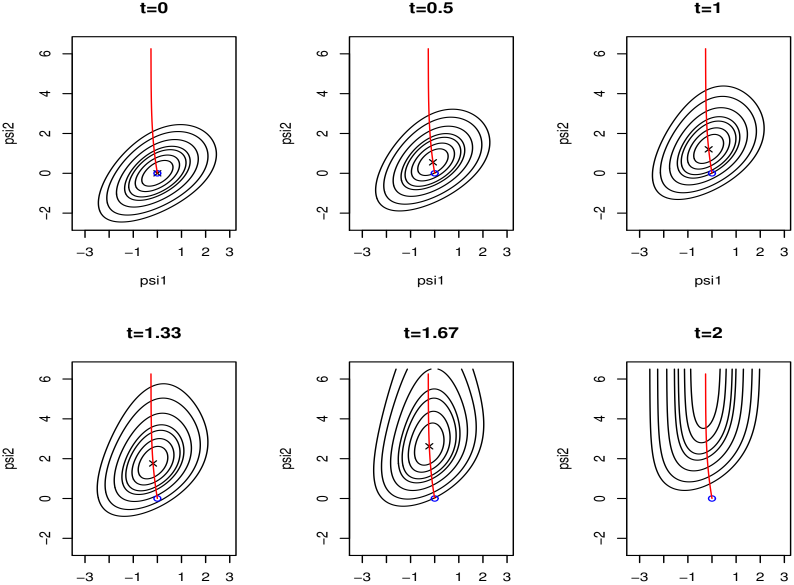
- line $s(t)$ from hypothesis, s_H, to data, s^0 : $s_H + t(s^0 - s_H)$
- $f(s;\psi_0)$ used to compute the probability at and beyond the observed s^0 ($t \geq 1$), conditional on being on the line $s(t)$.
- along the line $s(t)$ we have

  $$f(s;\psi_0)ds = f\{s(t);\psi_0\}dt = f\{s_H + t(s^0 - s_H);\psi_0\}dt\,.$$
- directional p-value:

$$p(\psi_0) = \frac{\int_1^{+\infty} t^{d_0-1} f\{s(t);\psi_0\}dt}{\int_0^{+\infty} t^{d_0-1} f\{s(t);\psi_0\}dt}$$

- one-dimensional integrals computed numerically

Log-likelihood along the line $s(t)$



Score variable?

- exponential family model

$$f(y;\theta) = \exp\{\varphi(\theta)' s(y) - c(\theta) - d(y)\}$$

- $f(s;\theta)$ available from saddlepoint approximation
- tangent exponential family model

$$f_{TEM}(s;\theta) = \exp\{\varphi(\theta;y^0)' s + \ell(\theta;y^0)\} h(s)$$

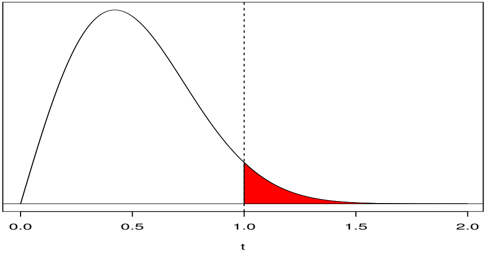
- saddlepoint approximation

$$f(s;\psi) \doteq \frac{e^{c/n}}{(2\pi)^d} \exp[\{(\psi - \hat{\psi})' s + \ell(\hat{\theta}) - \ell(\hat{\theta}_\psi)] |\hat{\jmath}_{\varphi\varphi}|^{-1/2}$$

- on line $s(t) = s_H + t(s^0 - s_H)$

Directional *p*-value

- The directional *p*-value is equal to 0.050



first order χ^2_2 approximation	$W(\psi_0)$	0.047
Skovgaard (2001 SJS) modified version	$W^*(\psi)$	0.048
simulated conditional		0.051

Testing independence in 2×3 contingency table

- contingency table on activity amongst psychiatric patients (Everitt, 1992 CH)

	Affective disorders	Schizophrenics	Neurotics
Retarded	12	13	5
Not retarded	18	17	25

- model: log-linear $y \sim$ Poisson, $\log\{E(y)\} = X\beta$
- H_0: independence
- nuisance parameter $\lambda \in \mathbb{R}^4$
- full model has an additional (ψ_1, ψ_2): interaction between the variables
- $H_0 : \psi = \psi_0 = (0, 0)$.

... 2×3 contingency table

- expected frequencies under the null hypothesis $t = 0$

	Affective disorders	Schizophrenics	Neurotics
Retarded	10	10	10
Not retarded	20	20	20

- need to stop at $t = t_{\max} = 2$.
- the expected frequencies corresponding to $t_{\max} = 2$

	Affective disorders	Schizophrenics	Neurotics
Retarded	14	16	0
Not retarded	16	14	30

- All tables along the line $s(t)$ have the same margins.

Another 2×3 table

- Consider the following data on party identification by race
  Agresti, 2002 Wiley

	Democrat	Independent	Republican
Black	103	15	11
White	341	105	405

- H_0: independence nested in saturated model
- first order likelihood ratio p-value: 2.43×10^{-20}
- directional p-value: 3.14×10^{-20}.

... party identification example

- Expected ($t = 0$)

	Democrat	Independent	Republican
Black	58.44	15.80	54.76
White	385.56	104.20	361.24

- Observed ($t = 1$)

	Democrat	Independent	Republican
Black	103	15	11
White	341	105	405

- Boundary ($t_{\max} = 1.251$)

	Democrat	Independent	Republican
Black	114.20	14.80	0.00
White	329.80	105.20	416.00

Log-linear models for contingency tables

- easier; already an exponential family model
- $y = (y_1, \ldots, y_C)$, $\quad X$ a $C \times d$ design matrix
- $E(y) = \exp(X\beta)$
- $$\ell(\beta) = \beta' X' y - \mathbf{1}' \exp(X\beta)$$
- $\varphi(\beta) = \beta$
- $\beta = (\lambda, \psi)$, and design matrix partitioned as $X = (X_1 \quad X_2)$
- $$\ell_\beta(\beta) = \begin{bmatrix} \ell_\lambda(\lambda, \psi) \\ \ell_\psi(\lambda, \psi) \end{bmatrix} = \begin{bmatrix} X_1'(y - e^{X\beta}) \\ X_2'(y - e^{X\beta}) \end{bmatrix}.$$
- constrained maximum likelihood estimate: $\ell_\lambda(\hat{\beta}_\psi) = X_1'(y - e^{X\hat{\beta}_\psi}) = 0$
- tangent exponential model = double saddlepoint distribution of $X_2' y$, given $X_1' y$

... details on log-linear models

- observed data point $s^0 = \mathbf{0}$
- expected value when $\psi = \psi_0$

$$s_H = -\ell_\beta(\hat\beta_{\psi_0}) = \begin{bmatrix} \mathbf{0} \\ -X_2^T(y - e^{X\hat\beta_{\psi_0}}) \end{bmatrix}.$$

- directional test goes radially from s_H towards the data point s^0 and beyond to the boundary in that direction.
- $f(s(t); \psi_0)$ along the line $s(t)$ computed using $\ell(\psi; t) = \ell(\hat\beta_\psi) + \hat\beta_\psi^T s(t)$.
- Nicola: use `glm` with numerical integration (univariate)

Infant survival data

	survival	gestation	smoking	age	Freq
1	No	<=260	<5	<30	50
2	Yes	<=260	<5	<30	315
3	No	>260	<5	<30	24
4	Yes	>260	<5	<30	4012
5	No	<=260	>5	<30	9
6	Yes	<=260	>5	<30	40
7	No	>260	>5	<30	6
8	Yes	>260	>5	<30	459
9	No	<=260	<5	>30	41
10	Yes	<=260	<5	>30	147
11	No	>260	<5	>30	14
12	Yes	>260	<5	>30	1594
13	No	<=260	>5	>30	4
14	Yes	<=260	>5	>30	11
15	No	>260	>5	>30	1
16	Yes	>260	>5	>30	124

...infant survival data

- Data (Agresti, 2002 Wiley) four dichotomous variables: age of mother (A), length of gestation (G), infant survival (I) and number of cigarettes smoked per day during gestation (S).
- response: length of gestation and infant survival
- null model: with all main effects and three first order interactions (IG, IA and SA) as the null model $\lambda \in \mathbb{R}^8$
- full model has two additional first order interaction parameters: IS and GA
- first order likelihood ratio *p*-value = 0.052.
- directional *p*-value = 0.056.

Conclusion

	scalar ψ	vector ψ
continuous response	r^* : $O(n^{-3/2})$	directional: $O(n^{-1})$
discrete response	r^* : $O(n^{-1})$	directional: $O(n^{-1})$

– tangent exponential model
– saddlepoint approximation
– easy, accurate

– extensions to nonlinear hypotheses, more complex models for categorical data...