Last week

- Nuisance parameters $f(y;\psi,\lambda)$, $\quad \ell(\psi,\lambda)$

- posterior marginal density
$$\pi_m(\psi) \doteq \frac{c}{\sqrt{(2\pi)^q}} e^{\ell_P(\psi)-\ell_P(\hat\psi)} |j_P(\hat\psi)|^{1/2} \frac{\pi(\psi,\hat\lambda_\psi)}{\pi(\hat\psi,\hat\lambda)} \frac{|j_{\lambda\lambda}(\hat\psi,\hat\lambda)|^{1/2}}{|j_{\lambda\lambda}(\psi,\hat\lambda_\psi)|^{1/2}}$$

- $\ell_p(\psi) = \ell(\psi,\hat\lambda_\psi)$

- exact conditional density
$f(y;\psi,\lambda) = \exp\{\psi^T s_1 + \lambda^T s_2 - k(\psi,\lambda)\}h(y)$

-
$$f(s_1 \mid s_2;\psi) \doteq \frac{c}{\sqrt{(2\pi)^q}} e^{\ell_p(\psi)-\ell_p(\hat\psi)} |j_p(\hat\psi)|^{-1/2} \frac{|j_{\lambda\lambda}(\hat\psi,\hat\lambda)|^{-1/2}}{|j_{\lambda\lambda}(\psi,\hat\lambda_\psi)|^{-1/2}}$$

... last week

Tail area approximation:

$$p(\psi) \doteq \Phi(r^*) = \Phi(r + \frac{1}{r}\log\frac{Q}{r}),$$

$$r = \pm\sqrt{[2\{\ell(\hat\theta) - \ell(\theta)\}]} \text{ or } \pm\sqrt{[2\{\ell_p(\hat\psi) - \ell_p(\psi)\}]}$$

$$Q = q_B = -\ell'(\theta)j^{-1/2}(\hat\theta)\frac{\pi(\hat\theta)}{\pi(\theta)} \quad \text{or}$$

$$-\ell_p(\hat\psi)j_p^{-1/2}(\hat\psi)\frac{\pi(\hat\psi,\hat\lambda)}{\pi(\psi,\hat\lambda_\psi)}\rho(\psi,\hat\psi)$$

$$Q = q = (\hat\theta - \theta)j^{1/2}(\hat\theta) \text{ or } (\hat\psi - \psi)j_p^{1/2}(\hat\psi)\rho^{-1}(\psi,\hat\psi)$$

$$\rho(\psi,\hat\psi) = \frac{|j_{\lambda\lambda}(\psi,\hat\lambda_\psi)|^{1/2}}{|j_{\lambda\lambda}(\hat\psi,\hat\lambda)|^{1/2}}$$

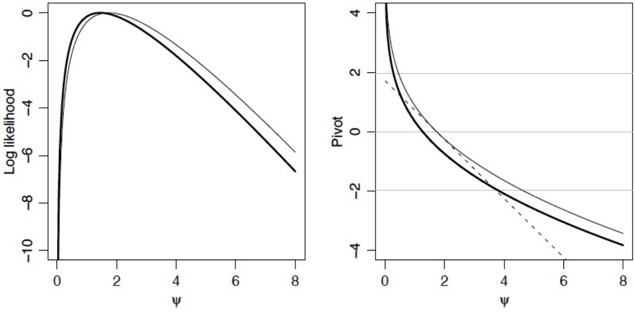


Figure 2.3: Inference for shape parameter ψ of gamma sample of size $n = 5$. Left: profile log likelihood ℓ_{p} (solid) and the log likelihood from the conditional density of u given v (heavy). Right: likelihood root $r(\psi)$ (solid), Wald pivot $t(\psi)$ (dashes), modified likelihood root $r^*(\psi)$ (heavy), and exact pivot overlying $r^*(\psi)$. The horizontal lines are at $0, \pm 1.96$.

This week

1. elimination of nuisance parameters and adjusted profile likelihood
2. approximate inference in location and transformation models
3. tail area approximations again
4. tangent exponential model
5. examples

Nuisance parameters

- $$\pi_{\mathrm{m}}(\psi) \doteq \frac{c}{\sqrt{(2\pi)^q}} e^{\ell_{\mathrm{p}}(\psi)-\ell_{\mathrm{p}}(\hat\psi)} j_{\mathrm{p}}^{1/2}(\hat\psi) \frac{\pi(\psi,\hat\lambda_\psi)}{\pi(\hat\psi,\hat\lambda)} \frac{|j_{\lambda\lambda}(\hat\psi,\hat\lambda)|^{1/2}}{|j_{\lambda\lambda}(\psi,\hat\lambda_\psi)|^{1/2}}$$
- $$= \frac{c}{\sqrt{(2\pi)^q}} e^{\ell_{\mathrm{A}}(\psi)-\ell_{\mathrm{A}}(\hat\psi)} j_{\mathrm{p}}^{1/2}(\hat\psi) \frac{\pi(\psi,\hat\lambda_\psi)}{\pi(\hat\psi,\hat\lambda)}$$
- $$\ell_{\mathrm{A}}(\psi) = \ell_{\mathrm{p}}(\psi) - \frac{1}{2}\log|j_{\lambda\lambda}(\psi,\hat\lambda_\psi)|$$
- $$f(s_1 \mid s_2;\psi) \doteq \frac{c}{\sqrt{(2\pi)^q}} |j_{\mathrm{p}}(\hat\psi)|^{-1/2} e^{\ell_{\mathrm{p}}(\psi)-\ell_{\mathrm{p}}(\hat\psi)} \left\{\frac{|j_{\lambda\lambda}(\hat\psi,\hat\lambda)|}{|j_{\lambda\lambda}(\psi,\hat\lambda_\psi)|}\right\}^{-1/2}$$
- $$= \frac{c}{\sqrt{(2\pi)^q}} |j_{\mathrm{p}}(\hat\psi)|^{-1/2} e^{\ell_{\mathrm{A}}(\psi)-\ell_{\mathrm{A}}(\hat\psi)}$$
- $$\ell_{\mathrm{A}}(\psi) = \ell_{\mathrm{p}}(\psi) + \frac{1}{2}\log|j_{\lambda\lambda}(\psi,\hat\lambda_\psi)|$$

Adjusted profile log-likelihood

- $\ell_{\mathsf{A}}(\psi) = \ell_{\mathsf{p}}(\psi) + A(\psi) = \ell(\psi, \hat{\lambda}_\psi) + A(\psi)$
- $A(\psi)$ assumed to be $O_p(1)$
- generic form is $A_{FR}(\psi) = +\frac{1}{2}\log|j_{\lambda\lambda}(\psi, \hat{\lambda}_\psi)| - \log|\frac{d(\lambda)}{d\hat{\lambda}_\psi}|$

  Fraser, 2003

- closely related $A_{BN}(\psi) = -\frac{1}{2}\log|j_{\lambda\lambda}(\psi, \hat{\lambda}_\psi)| + |\log\frac{d\hat{\lambda}}{d\hat{\lambda}_\psi}|$

  SM §12.4.1, BN 1983

- if $i_{\psi\lambda}(\theta) = 0$, then $\hat{\lambda}_\psi = \hat{\lambda} + O_p(n^{-1})$, suggesting we ignore last term
- if ψ is scalar, then in principle we can find a parametrization (ψ, λ) in which $i_{\psi\lambda}(\theta) = 0$

  SM §12.4.2

Log-likelihoods

- marginal $\ell_{\mathsf{m}}(\psi) = \log f_{\mathsf{m}}(t;\psi)$

$$f(y;\psi,\lambda) \propto f_{\mathsf{m}}(t;\psi) f_{\mathsf{c}}(s \mid t;\psi,\lambda)$$

- conditional $\ell_{\mathsf{c}}(\psi) = \log f_{\mathsf{c}}(s \mid t;\psi)$

$$f(y;\psi,\lambda) \propto f_{\mathsf{m}}(t;\psi,\lambda) f_{\mathsf{c}}(s \mid t;\psi)$$

- adjusted $\ell_{\mathsf{A}}(\psi) = \ell_{\mathsf{p}}(\psi) - \frac{1}{2}\log|j_{\lambda\lambda}(\psi,\hat{\lambda}_\psi)|$ $\qquad \lambda \perp \psi$

- refinements $\ell_{FR}(\psi)$, $\ell_{BN}(\psi), \ldots$

Sartori, 2003; Nov12 Prob 2.2

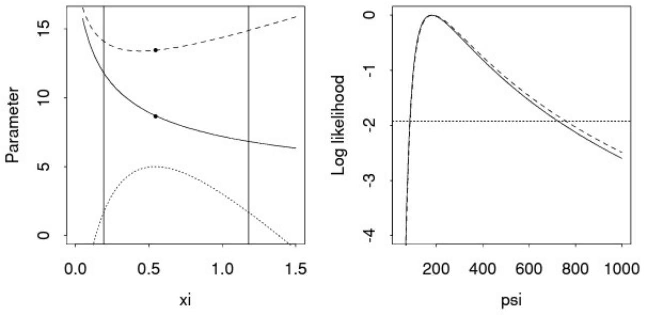


Figure 12.11 Likelihood analysis of Danish fire data. Left: variation of $\widehat{\sigma}_\xi$ (solid) and $\widehat{\lambda}_\xi$ (dashes) as a function of the shape parameter ξ. The blobs show the maximum likelihood estimates. The dotted line is the profile log likelihood $\ell_{\rm p}(\xi)$ and the vertical lines mark the limits of a 0.99 confidence interval for ξ. The orthogonal parameter $\widehat{\lambda}_\xi$ varies much less than does the non-orthogonal parameter $\widehat{\sigma}_\xi$ over the range of ψ considered. Right: profile log likelihood (solid) and modified profile log likelihood (dashes) for 0.99 quantile ψ of generalized Pareto distribution. The horizontal line determines the limits of a 0.95 confidence interval for ψ.

Example 12.25 (Generalized Pareto distribution) The expected information matrix for an observation with distribution function (6.38) is

$$i^*(\xi, \sigma) = \frac{1}{\sigma^2(1+\xi)(1+2\xi)} \begin{pmatrix} 2\sigma^2 & \sigma \\ \sigma & 1+\xi \end{pmatrix}.$$

Hence a parameter $\lambda = \lambda(\xi, \sigma)$ orthogonal to the shape parameter ξ satisfies the partial differential equation

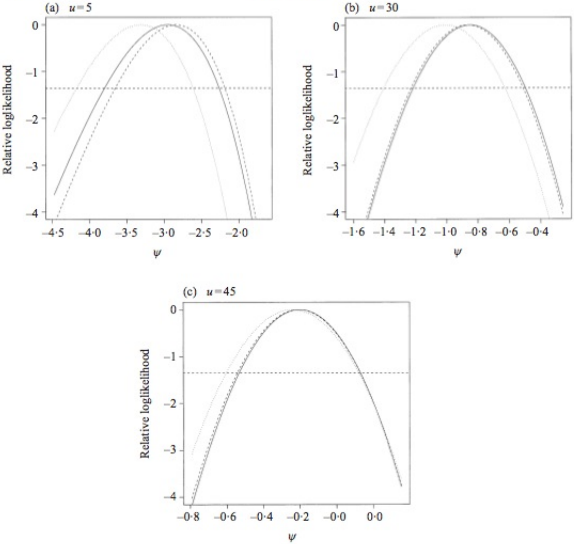


Fig. 1: Example 3. Inference about the common odds ratio in pairs of binomial observations. Relative loglikelihoods for ψ: profile (dotted), modified profile (dashed) and conditional (solid). The horizontal dashed line gives the 0·90 confidence interval. We consider $q = 100$, $m = 5$, $v_i = 3$ $(i = 1, \ldots, q)$, and three values of u.

Marginal inference

- conditional inference
  – linear exponential families   saddlepoint approx
- marginal inference?
- Bayesian posterior   Laplace approx
- transformation models

- exponential families – non-linear parameter of interest
- tangent exponential models
- everything else

Location model

- location model $f(y_i; \theta) = f_0(y_i - \theta), \quad y_i, \theta \in \mathbb{R}$
- $f(y; \theta) =$
- $\ell(\theta; y) =$
- change of variable $(y_1, \ldots, y_n) \leftrightarrow (\hat{\theta}, a_1, \ldots, a_n)$
- $a_i =$
- $f(\hat{\theta}, a; \theta) =$

... location models

- $f(\hat{\theta}, a; \theta) =$

- $f(a; \theta) =$

- $f(\hat{\theta} \mid a; \theta) =$

Approximate p-values

$$\int^{\hat\theta} f(\hat\vartheta \mid a; \vartheta) d\hat\vartheta = c\int^{\hat\theta} |j(\hat\theta)|^{1/2} e^{\ell(\theta;\hat\theta,a)-\ell(\hat\theta;\hat\theta,a)} d\hat\theta$$

$$= \int^{r} c e^{-\frac{1}{2}r^2} |j(\hat\theta)|^{1/2} \frac{dr}{\ell_{;\hat\theta}(\hat\theta;\hat\theta,a) - \ell_{;\hat\theta}(\theta;\hat\theta,a)} d\hat\theta$$

$$= \int^{r} e^{-\frac{1}{2}r^2} \frac{r}{q} dr$$

$$= \Phi(r + \frac{1}{r}\log\frac{q}{r})$$

$$= \{\ell_{;\hat\theta}(\hat\theta;\hat\theta,a) - \ell_{;\hat\theta}(\theta;\hat\theta,a)\} j(\hat\theta)^{-1/2}$$

$$= \ell_\theta(\theta) j(\hat\theta)^{-1/2}$$

location model

... approximate p-values

Nuisance parameters

- Regression-scale models: $y_i = x_i^T \beta + \sigma \epsilon_i, \quad \epsilon_i \sim f_0(\cdot)$
- $f(y; \beta, \sigma^2) = f_c(\hat{\beta}, \hat{\sigma} \mid a; \sigma, \beta) f_m(a) \quad a =$
- parameter of interest $\hat{\beta}_j$, say
- $t_j = \dfrac{\hat{\beta}_j - \beta_j}{v_j}$ has marginal density free of $\beta_{(j)}, \sigma$
- $t_{p+1} = \log \hat{\sigma} - \log \sigma$ has marginal density free of β
- $v_j = Var(\hat{\beta}_j) =$
- p-value functions using Laplace approximation again

These are all special cases

- to compute p-value we need to integrate a density approximation
- this density approximation is to either a marginal or a conditional density
- the integration involves the derivative of the log-likelihood, with respect to the data
- simplify the density approximation to incorporate only this first derivative
- use the resulting simpler form as the basis for approximate likelihood based inference

Tangent exponential model

- $f_{TEM}(s;\theta) = c\, e^{\ell\{\varphi(\theta);y\}+\varphi(\theta)^T s}\, |j_{\varphi\varphi}\{\hat\varphi(s);s\}|^{-1/2}$
- p-value = $\Phi(r^*) = \Phi(r + \frac{1}{r}\log\frac{Q}{r})$
- $Q = \{\varphi(\hat\theta) - \varphi(\theta)\}\jmath_{\varphi\varphi}^{-1/2}(\hat\varphi) = \dfrac{\varphi(\hat\theta)-\varphi(\theta)}{\varphi_\theta(\hat\theta)}\jmath_{\theta\theta}^{1/2}(\hat\theta)$
- $Q = \dfrac{|\varphi(\hat\theta) - \varphi(\hat\theta_\psi) \quad \varphi_\lambda(\hat\theta_\psi)|}{|\varphi_\theta(\hat\theta)|}\dfrac{|j(\hat\theta)|^{1/2}}{|j_{\lambda\lambda}(\hat\theta_\psi)|^{1/2}}$

.. tangent exponential model

- $$\varphi(\theta) = \ell_{;V}(\theta; y) = \sum_{i=1}^{n} \frac{\partial \ell(\theta; y)}{\partial y_i} V_i$$

- $$V_i = -\left(\frac{\partial z_i}{\partial y_i}\right)^{-1} \frac{\partial z_i}{\partial \theta}\bigg|_{(\hat{\theta}, y)}$$

BDR, Ch. 8.4, 8.5

Example: top quark

- model $Y \sim \text{Poisson}\,(\mu + b)$, $\quad b$ known
- data $y = 27 \quad b = 6.7$
- mid p-value $\quad \text{Pr}(Y > 27) + \frac{1}{2}\text{Pr}(Y = 27) = 18.45 \times 10^{-10}$
- approximation: $\Phi(r^*) = \Phi\left(r + \dfrac{1}{r}\log\dfrac{q}{r}\right) = \; 15.85 \times 10^{-10}$
- continuity correction: $\quad \Phi\{r^*(y + \frac{1}{2})\} = \; 32.36 \times 10^{-10}$
  $\text{Pr}(Y \geq 27) \; = \; 29.83 \times 10^{-10}$

Abe et al., 1995

... top quark

	$\text{pr}(Y \geq y^0; \mu = 0)$	$\text{pr}(Y > y^0; \mu = 0)$
Exact	29.83×10^{-10}	7.06×10^{-10}
Mid-p	18.45×10^{-10}	
$\Phi(r^*)$	15.85×10^{-10}	
$N(\theta, \theta)$	4.45×10^{-4}	2.21×10^{-5}
$N(\theta, \hat{\theta})$	1.02×10^{-4}	4.68×10^{-5}

Two-sample comparison

data on cost of treatment: standard vs new treatment

Group 1	30	172	210	212	335	489	651	1263
	1875	2213	2998	4935				
Group 2	121	172	201	214	228	261	278	279
	561	622	694	848	853	1086	1110	1243

model $Y_{1i} \sim \mathrm{Exp}(\mu_1), \quad Y_{2i} \sim \mathrm{Exp}(\mu_2) \quad \psi = \mu_1/\mu_2$

Exact inference $\left(\frac{\bar{Y}_{1.}}{\mu_1}\right) / \left(\frac{\bar{Y}_{2.}}{\mu_2}\right) \sim F_{2n,2m}$

exact 95% confidence interval for ψ: $(0.98, 4.19)$
approx 95% confidence interval for ψ: $(0.98, 4.185)$

Evans et al. 1999

Logistic regression

The first ten out of 79 sets of observations on the physical characteristics of urine. Presence/absence of calcium oxalate crystals is indicated by 1/0. Two cases with missing values.

Case	Crystals	Specific gravity	pH	Osmolarity	Conductivity	Urea	Calcium
1	0	1.021	4.91	725	—	443	2.45
2	0	1.017	5.74	577	20.0	296	4.49
3	0	1.008	7.20	321	14.9	101	2.36
4	0	1.011	5.51	408	12.6	224	2.15
5	0	1.005	6.52	187	7.5	91	1.16
6	0	1.020	5.27	668	25.3	252	3.34
7	0	1.012	5.62	461	17.4	195	1.40
8	0	1.029	5.67	1107	35.9	550	8.48
9	0	1.015	5.41	543	21.9	170	1.16
10	0	1.021	6.13	779	25.7	382	2.21
⋮							⋮

Andrews & Herzberg, 1985

Model: Independent binary responses $Y_1, \ldots, Y_n$ with

$$\Pr(Y_i = 1) = \frac{\exp(x_i^T \beta)}{1 + \exp(x_i^T \beta)}$$

Fitting generalized linear model in R:

```
data(urine)
fit <- glm(r~gravity+ph+osmo+cond+urea+calc,
family = binomial, data=urine)
summary(fit)
              Estimate Std. Error z value Pr(>|z|)
(Intercept) -355.33771  222.76696  -1.595  0.11069
gravity      355.94379  222.11004   1.603  0.10903
ph            -0.49570    0.56976  -0.870  0.38429
osmo           0.01681    0.01782   0.944  0.34536
conduct       -0.43282    0.25123  -1.723  0.08493 .
urea          -0.03201    0.01612  -1.986  0.04703 *
calc           0.78369    0.24216   3.236  0.00121 **
```

A closer look at coefficient of `urea`

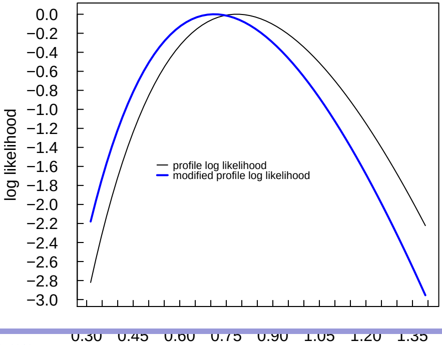
method	lower bound	upper bound	p-value for 0
$\Phi(q)$	-0.063	-0.0006	0.047
$\Phi(r)$	-0.067	-0.0025	0.033
$\Phi(r^*)$	-0.058	0.0002	0.052

```
library(cond) # part of package 'hoa' on cran-r

urine.cond.urea <- cond.glm(urine.glm,offset=urea)
> summary(urine.cond.urea,test=0)

...

Test statistics
---------------
 hypothesis : coef( urea ) = 0
                                             statistic   tail prob.
Wald pivot                                      -1.986      0.02351
Wald pivot (cond. MLE)                          -1.852      0.03202
Likelihood root                                 -2.133      0.01648
Modified likelihood root                        -1.925      0.02713
Modified likelihood root (cont. corr.)          -1.917      0.02760
```

A closer look at coefficient of `urea`

method	lower bound	upper bound	p-value for 0
$\Phi(q)$	-0.063	-0.0006	0.047
$\Phi(r)$	-0.067	-0.0025	0.033
$\Phi(r^*)$	-0.058	0.0002	0.052

Profile and modified profile log likelihoods



Several 2 × 2 tables.

Institution	y_1	m_1	y_2	m_2	Institution	y_1	m_1	y_2	m_2
1	3	4	1	3	12	2	2	0	2
2	3	4	8	11	13	1	4	1	5
3	2	2	2	3	14	2	3	2	4
4	2	2	2	2	15	2	4	4	6
5	2	2	0	3	16	4	12	3	9
6	1	3	2	3	17	1	2	2	3
7	2	2	2	3	18	3	3	1	4
8	1	5	4	4	19	1	4	2	3
9	2	2	2	3	20	0	3	0	2
10	0	2	2	3	21	2	4	1	5
11	3	3	3	3					

Lipsitz et al. 1988: Biometrics

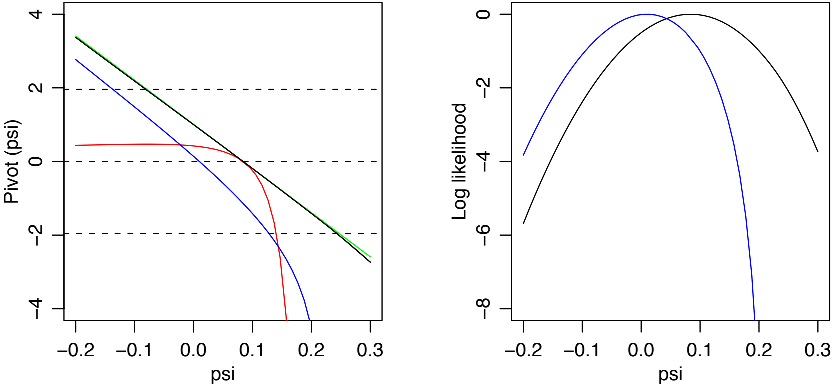
... matched pairs

Model: $Y_{1i} \sim \text{Binomial}(m_{1i}, p_{1i}) \quad Y_{2i} \sim \text{Binomial}(m_{2i}, p_{2i})$

parameter of interest $\psi = p_{2i} - p_{1i}$
nuisance parameters $p_{1i}, i = 1, \dots 21$

inference for ψ :

	lower	upper	point estimate	p-value for $\psi = 0$
$\Phi(r)$	-0.081	0.243	0.10	0.16
$\Phi(r^*)$	-0.137	0.126	0.01	0.45



likelihood root, r^*, q

profile log likelihood,
modified profile

Example G: Cox & Snell, 1980

	cost	date	T1	T2	cap	PR	NE	CT	BW	N	PT
1	460.05	68.58	14	46	687	0	1	0	0	14	0
2	452.99	67.33	10	73	1065	0	0	1	0	1	0
3	443.22	67.33	10	85	1065	1	0	1	0	1	0
4	652.32	68.00	11	67	1065	0	1	1	0	12	0
5	642.23	68.00	11	78	1065	1	1	1	0	12	0
6	345.39	67.92	13	51	514	0	1	1	0	3	0
7	272.37	68.17	12	50	822	0	0	0	0	5	0
8	317.21	68.42	14	59	457	0	0	0	0	1	0
9	457.12	68.42	15	55	822	1	0	0	0	5	0
10	690.19	68.33	12	71	792	0	1	1	1	2	0
11	350.63	68.58	12	64	560	0	0	0	0	3	0
12	402.59	68.75	13	47	790	0	1	0	0	6	0
13	412.18	68.42	15	62	530	0	0	1	0	2	0
14	495.58	68.92	17	52	1050	0	0	0	0	7	0
15	394.36	68.92	13	65	850	0	0	0	1	16	0

$n = 32, d = 8$

Linear regression, non-normal error

- Model $Y_i = \beta_0 + x_i^T \beta + \sigma \epsilon_i$
- $\epsilon \sim N(0, 1)$ or $\epsilon \sim t_\nu$

	Normal		t_4, first order		t_4, third order	
	Est (SE)	z	Est (SE)	z	Est (SE)	z
Constant	−13.26 (3.140)	−4.22	−11.30 (3.67)	−3.01	−11.86 (3.70)	−3.21
`date`	0.212 (0.043)	4.91	0.191 (0.048)	3.97	0.196 (0.049)	4.02
`log(cap)`	0.723 (0.119)	6.09	0.648 (0.113)	5.71	0.682 (0.129)	5.31
`NE`	0.249 (0.074)	3.36	0.242 (0.077)	3.12	0.239 (0.080)	2.97
`CT`	0.140 (0.060)	2.32	0.144 (0.054)	2.68	0.143 (0.063)	2.26
`log(N)`	−0.088 (0.042)	−2.11	−0.060 (0.043)	−1.40	−0.072 (0.048)	−1.51
`PT`	−0.226 (0.114)	−1.99	−0.282 (0.101)	−2.80	−0.265 (0.110)	−2.42

ν	`log(N)`		`PT`	
	First order	Third order	First order	Third order
4	0.162	0.151	0.005	0.024
6	0.110	0.116	0.007	0.032
8	0.081	0.098	0.009	0.036
10	0.064	0.086	0.011	0.038
20	0.036	0.064	0.016	0.045
40	0.025	0.053	0.029	0.050
100	0.020	0.047	0.022	0.053
∞	0.035	0.045	0.046	0.057

```
library(marg)
# part of package 'hoa' on cran-r
data(nuclear)

# Fit normal-theory linear model and examine its contents:

nuc.norm <- lm( log(cost) ~ date + log(cap) + NE + CT + log(N) + PT,
+                  data = nuclear )
summary(nuc.norm)

# Fit linear model with t errors and 4 df and examine its contents:

nuc.t4 <- rsm( log(cost) ~ date + log(cap) + NE + CT + log(N) + PT,
+                  data = nuclear, family = student(4) )
summary(nuc.t4)
plot(nuc.t4)

# Conditional analysis for partial turnkey guarantee:

nuc.t4.pt <- cond( nuc.t4, offset = PT )
summary(nuc.t4.pt)
plot(nuc.t4.pt)

# For conditional analysis for other covariates, replace pt by
# log(N), ...
```

Type II censored data

40 units on test, 28 failures at (log) times

0.0507	0.0579	0.0784	0.0954	0.1376	0.2249	0.2362	0.2481
0.2501	0.2811	0.3027	0.3091	0.4296	0.5379	0.5621	0.5781
0.7811	0.8228	0.9455	0.9871	1.0060	1.0335	1.0377	1.0471
1.0876	1.2473	1.2776	1.3445				

Weibull model: $f(y;\mu,\sigma) = e^{(y-\mu)/\sigma}\exp\{-e^{(y-\mu)/\sigma}\}$

90% confidence intervals

	μ	σ
$\Phi(r)$	$(-0.116, 0.476)$	$(0.700, 1.217)$
$\Phi(r^*)$	$(-0.107, 0.510)$	$(0.743, 1.320)$
Exact (num. int.)	$(-0.11, 0.51)$	$(0.724, 1.277)$

Lawless 2003 Ch.5; Wong & Wu 2003

Vector parameter of interest

- use tangent exponential model (or usual exponential family model)
- construct a scalar parameter of interest representing direction in sample space
- apply higher order approximation
- Example $y \sim N(\mu, \Sigma)$; $H_0 : \Sigma^{-1}$ is tri-diagonal
- First-order Markov dependence in a graphical model

Nominal (%)	1.0	2.5	5.0	10.0	25.0	50.0	75.0	90.0	95.0	97.5
First order	5.5	10.5	17.0	27.0	48.7	73.0	89.5	96.7	98.5	99.4
Second order	1.1	2.6	5.0	10.1	24.8	49.8	74.9	89.9	94.9	97.4