LTCC/Reid:

Homework Problems for Lecture of 19/11/12; due 26/11/12

1. Suppose $Y_1, \dots, Y_n$ are independent and identically distributed from a model $f(y;\theta), y \in R, \theta \in R$, and that $\pi(\theta)$ is a proper prior density (with respect to Lebesgue measure on R). Denote by $\hat{\theta}_\pi$ the posterior mode:

$$\hat{\theta}_\pi = \arg\sup_\theta \pi(\theta \mid y)$$

which we assume is obtained as the unique root of the equation

$$\frac{d}{d\theta} \log \pi(\hat{\theta}_\pi \mid y) = 0. \qquad (1)$$

Denote by $\tilde{\theta}$ the posterior mean:

$$\tilde{\theta} = \int \theta \pi(\theta \mid y) d\theta.$$

Show that

$$\hat{\theta}_\pi - \hat{\theta} = O_p(\frac{1}{n}), \text{ and } \tilde{\theta} - \hat{\theta} = O_p(\frac{1}{n}),$$

where $\hat{\theta}$ is the maximum likelihood estimator of θ.

2. Consider a linear regression model

$$y_i = x_i^T \beta + \epsilon_i, \quad i = 1, \dots, n$$

where x_i and β are $p \times 1$ vectors, and ϵ_i are i.i.d. $N(0, \sigma^2)$. Compare the log-likelihood ratio statistics for inference about β, based on the

(a) profile log-likelihood $w(\beta) = 2\{\ell_{\rm p}(\hat{\beta}) - \ell_{\rm p}(\beta)\}$,

(b) adjusted profile log-likelihood $w_{\rm A}(\beta) = 2\{\ell_{\rm A}(\hat{\beta}_{\rm A}) - \ell_{\rm A}(\beta)\}$, and

(c) modified profile log-likelihood $w_{\rm M}(\beta) = 2\{\ell_{\rm M}(\hat{\beta}_{\rm M}) - \ell_{\rm M}(\beta)\}$,

where

$$\ell_{\rm p}(\beta) = \ell(\beta, \hat{\sigma}^2_\beta), \quad \ell_{\rm A}(\beta) = \ell_{\rm p}(\beta) - \frac{1}{2}\log|j_{\sigma^2\sigma^2}(\beta, \hat{\sigma}^2_\beta)|, \text{ and } \ell_{\rm M}(\sigma^2) = \ell_{\rm A}(\beta) + \log|\frac{d\hat{\sigma}^2}{d\hat{\sigma}^2_\beta}|,$$

and $\hat{\beta}_{\rm A}$, $\hat{\beta}_{\rm M}$ are the adjusted and modified maximum likelihood estimators, respectively.