Continuous time processes - Part I

We are going to look at \( \{X(t) \mid t \geq 0\} \) such that the state space is \( \{0, 1, 2, \ldots\} \) and such that \( \forall t_0 < t_1 < t_2 < \ldots \) \( \{X(t_2)\} \) is a MC. We also assume time homogeneous processes. The Markov property is \( \Rightarrow \) given \( X(t) \), \( \{X(s) \mid s \prec t\} \) and \( \{X(s) \mid s \succ t\} \) are independent.

A typical "data set" or sample path or realization looks like:

\[
x(t)
\]

\[
\begin{array}{c}
\text{form a point process}
\end{array}
\]
A discrete-time Markov chain \( \{X(t) : t \geq 0\} \) may be thought of as a point process where points occur in the following way:
- \( X(t) \) stays in state \( i \) for a time \( T_i \).
- \( T_i \) is exponential (\( \mathcal{E}(\lambda_i) \)).
- The \( T_i \) are independent.
- When leaving \( i \), the probability of going to \( j \) is \( P_{ij} \) (\( j \neq i \)).

Remarks:
(a) If the process is observed only at jumps, then we get a discrete-time Markov chain.
(b) A semi-Markov process would be similar except that the times between jumps would not be exponential and the time stayed in \( i \) would depend on the target \( j \).
(c) \( P(T_i > t+s | T_i > s) = P(T_i > t) \) 

\[ \text{Lack of memory} \]

© © © of the Markov Property.

© © © \( T_i \sim \text{exponential}(\beta_i) \)

Note that large rates \( \Rightarrow \) small means so it's not hard to imagine jumping right out of the state space.
The $q_i$ may be thought of as the rate of leaving $i$. Typically $0 < q_i < \infty$ but $q_i = 0$ may be thought of as an absorbing state and $q_i = \infty$ an "instantaneous" state—leave as soon as you enter.

The transition rate from $i$ to $j \neq i$ is defined as

$$q_{ij} = q_i \cdot P_{ij}$$

The process is regular or honest if there can only be a finite number of transitions in $[0, t]$, $\forall t > 0$. In fact, a possibility which we will see in a Birth process. Assuming none doesn't escape the state space

$$\sum_{j \neq i} P_{ij} = 1$$
Set \( q_{ii} = -q_i \) and \( Q = \{ q_{ij} \} \).

This matrix is called the generator of the process and plays a similar role to that of the transition matrix \( P \) in a discrete time MC. Notice

\[
\sum_j q_{ij} = \left( \sum_{j \neq i} q_i P_{ij} \right) - q_i = \left( q_i \sum_{j \neq i} P_{ij} \right) - q_i = 0
\]

That is

\[ Q \mathbf{1} = 0 \]
The CKE are
\[ P(t+h) = P(t) P(h), \]
where \( P(t) = \{ p_{ij}(t) \} \) and \( p_{ij}(t) \) is the transition function.

Consider
\[ p_{ij}(t+h) = \sum_k P(X(t+h) = j | X(h) = k, X(0) = i) \]
\[ \times P(X(h) = k | X(0) = i) \]
\[ = \sum_k P(X(t+h) = j | X(h) = k) P(X(h) = k | X(0) = i) \]

Note that we have conditioned back to \( X(h) \). So
\[ p_{ij}(t+h) = \sum_k p_{kj}(t) p_{ik}(h) \]
\[ = \sum_k p_{ik}(h) p_{kj}(t) \]

which is, of course, restating the CKE in component form. In matrix terms
\[ P(t+h) = P(h) P(t) \]
Now, under certain conditions,

\[ Q = \lim_{h \to 0} \frac{P(h) - I}{h} = \dot{P}(0) \]

In this case

\[ \frac{P(t+h) - P(t)}{h} = \frac{P(h) P(t) - P(t)}{h} \]

\[ = \left[ \frac{P(h) - I}{h} \right] P(t) \]

so that

\[ \dot{P}(t) = Q P(t) \quad \Leftrightarrow \quad P(t) = e^{Q t} \]

which are the backward equations.

There was an interchange of limit and summation which has to be justified.
For \( \{ X(t) \mid t \geq 0 \} \) the KBE and KFE's are

\[
\dot{P}(t) = Q P(t)
\]

\[
\dot{P}(t) = P(t) G
\]

\[
\rho_{ij}(t) = \sum_k Q_{ik} P_{kj}(t)
\]

\[
\rho_{ij}(t) = \sum_k P_{ik}(t) Q_{kj}
\]

\[
Q = \{ q_{ij} \} \text{ is often denoted by } \Lambda \text{ or } G.
\]

\[
i \neq j \quad \rho_{ij}(h) = q_{ij} \cdot h + o(h)
\]

\[
q_{i} = -q_{ii} \quad Q \rho = 0 - \text{often}
\]
To see this go back to the component equation

\[ p_{ij}(t+h) - p_{ij}(t) = (\sum_{k} p_{ik}(h) p_{kj}(z)) - p_{ij}(t) \]

\[ = (\sum_{k \neq i} p_{ik}(h) p_{kj}(z)) + p_{ic}(h) p_{ij}(t) - p_{ij}(t) \]

or

\[ \frac{p_{ij}(t+h) - p_{ij}(t)}{h} = \sum_{k \neq i} \frac{p_{ik}(h)}{h} p_{kj}(z) \]

\[ + \frac{(p_{ic}(h)-1) p_{ij}(t)}{h} \]

Now \( \frac{p_{ik}(h)}{h} \rightarrow \varphi_{ik} \)

\( \frac{p_{ic}(h)-1}{h} \rightarrow \varphi_{ii} \)

\( -\varphi_{i} \)

must be proved

Assuming we can interchange the \( \lim \) and \( \sum \) we get

\[ \dot{p}_{ij}(t) = (\sum_{k \neq i} \varphi_{ik} p_{kj}(z)) - \varphi_{i} p_{ij}(t) \]
Examples

Poisson process of rate $\lambda$ on $t \geq 0$

- $N(0) = 0$
- $N(t)$ has independent increments
- $N(t)$ has "local" transitions given by

$$P(N(t+h) = j+1 | N(t) = j) = \lambda h + o(h)$$
$$P(N(t+h) > j+1 | N(t) = j) = o(h)$$
$$P(N(t+h) = j | N(t) = j) = 1 - \lambda h + o(h)$$

This leads to $N(t) \sim \text{Poisson}(\lambda t)$.

We can specify the Poisson model by assuming that the only possible transitions out of state $i$ is to

$i+1$ with constant rate $\lambda$. Then

we have the KE's

$$\dot{P}_{ik}(t) = \lambda \left( P_{ik-1}(t) - P_{ik}(t) \right) \quad (E)$$
$$\dot{P}_{ik}(t) = \lambda \left( P_{i+1,k}(t) - P_{ik}(t) \right) \quad (B)$$
The sol'n to either is

\[ P_{kn}(t) = e^{-\lambda t} (\lambda t)^{n-k} \frac{1}{(k-i)!} \quad k \geq i \]

Of course \( P_{kn}(t) = 0 \) for \( k < i \). We can solve these use generating functions

\[ P_{\ell}(z, t) = \sum_{k=0}^{\infty} P_{kn}(t) z^k \]

The (F) and (B) become

\[ \dot{P}_{\ell}(z, t) = \lambda (z-1) P_{\ell}(z, t) \quad \text{(F)} \]

\[ \dot{P}_{\ell}(z, t) = \lambda (P(z, t) - P_{\ell}(z, t)) \quad \text{(B)} \]

with initial conditions

\[ P_{\ell}(z, 0) = z^i \]

The (F) eq's have sol'n

\[ P_{\ell}(z, t) = z^i e^{\lambda t (z-1)} \quad \text{(x)} \]
For our Poisson process \( i = 0 \) 
\( \Rightarrow N(0) = 0 \) which yields the 
PDF 
\[ e^{-t} (t-1) \] 
for \( N(t) \). This is the PDF of a Poisson(\( \lambda t \)).

You can verify that (*) satisfies the 
(3) eq'ns.
Birth + Death process

\[ i \rightarrow i+1 \quad \text{with rate } \lambda_i \]
\[ i \rightarrow i-1 \quad \text{" } \quad \mu_i \quad \text{" death rate} \]

The simple birth process (Rule) has \( \lambda_i = i \) and \( \mu_i = i \). If we set \( \lambda_i = \lambda + \mu \), \( \mu_i = \mu \)
yields a simple B + D + immigration process. Set

\[ P_i(z,t) = E(z^{N(t)+s} \mid N(0) = i) \]
\[ = E(z^{N(t)} \mid N(0) = i) \]

Then the KFE is

\[ \dot{P}_i = (\lambda z - \mu) (z-1) \frac{d}{dz} P_i + \gamma (z-1) P_i \]
while the KBE are
\[
\frac{dP_i}{dt} = (\lambda + \gamma)(P_{i+1} - P_i) + \mu i (P_i - P_{i-1})
\]
The PDE for the KBE can be solved but here the KBE is simpler.

\(N(t)\) is made up of a component related to the initial ancestors and a contribution from immigration (there are \(u(\gamma)\)). So
\[
P_i(z, t) = A(z, t) B(z, t)
\]
where \(A(z, t)\) is the pgf of the component derived from immigration and \(B(z, t)\) that from an initial ancestor. Substitute into the KBE to get
\[
\dot{\bar{A}} = \gamma (B-1) \bar{A}, \quad \bar{B} = (\lambda B - u)(B-1)
\]

\(A(0) = 1, \ B = \frac{z}{B}\) for the initial conditions \(t=0\).

\[\lambda < \gamma (B-1)\]
\[ B(z, t) = \frac{\lambda (1-z) e^{(1-\mu) t} - (\mu - 1) z}{\lambda (1-z) e^{(1-\mu) t} - (\mu - 1) z} \]

\[ A(z, t) = \left[ \frac{\lambda (1-z) e^{(1-\mu) t} - (\mu - 1) z}{\lambda - \mu} \right] ^{-1/\lambda} \]

If \( \lambda = 0 \) then

\[ P(N(t) = 0 | N(0) = i) = B(0, i) \]

\[ = \left( \frac{\mu e^{(1-\mu) t} - \mu}{\lambda e^{(1-\mu) t} - \mu} \right) ^i \]