For discrete time (\( T = \mathbb{Z}^+ \)) the stationary distribution \( \pi' \) is any \( \mathbf{P} \) satisfying \( \pi' = \pi' \mathbf{P} \).

Note - \( \pi' \) is not unique.

We know:
- \( j \) is recurrent iff \( P_{jj}(1) = \infty \) (i.e., \( \sum_{m} P_{jj}(m) = \infty \))
- \( j \) recurrent then it is positive recurrent iff
  \[ \pi'_j = \frac{\lim_{\alpha \to 1} (1-\alpha) P_{jj}(\alpha)}{\mathbb{E}(T_{jj})} \]
  and then \( \pi'_j = \frac{1}{\mathbb{E}(T_{jj})} \)

Notice that
\[
\frac{1 - F_{jj}(\alpha)}{1 - \alpha} = 1 - \mathbb{E}(\alpha^{T_{jj}}) = \frac{1 - \sum_{k} \alpha^k P(T_{jj} = k)}{1 - \alpha}
\]

\[= \left[ \sum_{k} P(T_{jj} = k) - \sum_{k} \alpha^k P(T_{jj} = k) \right] / (1 - \alpha)\]
\[= \sum_{k} P(T_{jj} = k) \frac{1 - \alpha^k}{1 - \alpha} \]
- process irreducible & positive recurrent
  \[ \Rightarrow \pi \text{ is unique} \text{ & } \pi > 0 \]
  \[ \pi_i = \frac{1}{E(T_{ii})} \]

If the process is irreducible with a countably infinite state space then
\( \pi \) exists \( \Rightarrow \) positive recurrent

If the process is irreducible & finite then all states are positive recurrent

Irreducible & positive recurrent & aperiodic
\[ \Rightarrow \lim_{m \to \infty} p_{ij}(m) = \pi_{ij} > 0 \]
\[ \frac{\pi}{\pi} \text{ is the unique stationary distribution} \]

For irreducible & aperiodic & null recurrent we do not have a stationary distribution. (By applying renewal theory we in fact know \( p_{ij}(m) \to 0 \).) From the coupling argument we still have
\[ p_{ik}(m) - p_{\pi k}(m) \to 0 \quad \forall i,j,k \quad (*) \]

We now use this to show \( p_{ij}(m) \to 0 \).
Now suppose $p_{ij}(m)$ does not go to zero as $m \to \infty$. For fixed $i \neq j$, if a subsequence of the $m$'s so that $p_{ij}(m_k) \to \alpha_j$ and $\alpha_j$ will not depend on $i$.  Of $(*)$ but the subsequence might. Now let $\epsilon_i > 0$.

For each $i$ choose $m_k$ so that $m_k \in \{ m^{(i)} \}$ and $p_{ij}(m_k)$ is within $\epsilon_i$ of $\alpha_j$. In addition choose the $m_k$'s to increase with $i$. We then have

$$p_{ij}(m_k) \to \alpha_j \quad \text{for each } j$$

and not all $\alpha_j$ are 0. (We have just employed a diagonalization argument.)

Let $F$ be a finite set of $j$ states. Then

$$\sum_{j \in F} \alpha_j = \lim_{k \to \infty} \sum_{j \in F} p_{ij}(m_k) \leq 1$$

$$\leq \sum_{j \in 0} p_{ij}(m_k) = 1$$

Hence $0 < \sum_{j \in 0} \alpha_j \leq 1$.

Now

$$\sum_{j \in F} p_{ij}(m_k) p_{ij} \leq p_{ij}(m_{k+1}) = \sum_{j \in F} p_{ij} p_{ij}(m_k) \quad \to \alpha_j$$
\[ \sum_{l} x_{l} P_{l,j} \leq x_{j} \]

\[ \sum_{l} x_{l} P_{l,j} = x_{j} \quad \forall j \]

In fact equality must hold \( \forall j \) for if we had \( < \) for some \( j \) then

\[ \sum_{l} x_{l} = \sum_{l} x_{l} \sum_{j} P_{l,j} = \sum_{j} x_{j} P_{l,j} = \sum_{j} x_{j} P_{l,j} < \sum_{j} x_{j} \]

which is impossible.

\[ \sum_{l} x_{l} P_{l,j} = x_{j} \quad \forall j \]

Now set \( \alpha = \sum_{j} x_{j} + \frac{\pi_{j}}{\alpha} \). We then have \( \pi \) is a \( \text{PF} + \pi' = \alpha \cdot \pi' \). But this cannot be. It follows that

\[ p_{i,j}(m) \to 0 \quad \forall i, j \]

Irreducible + aperiodic + transient - obvious

that \( p_{i,j}(m) \to 0 \)

**Theorem** MC irreducible + aperiodic then

\[ p_{i,j}(m) \to \frac{1}{E(T_{i,j})} \]
Assume irreducible + aperiodic

Let $i, j \in S$. We want to show $p_{ij}(m) > 0$ for $m$ large enough.

Since $j$ is aperiodic $\exists m_1, \ldots, m_n$ with $\gcd = 1$ and $p_{jj}(m_i) > 0$ for $1 \leq i \leq n$. Now $\exists M$ such that for $m \geq M$ we can write $m = \sum_{a \in \mathbb{Z}^+} q_a m^a$.

Now use the CKE to get

$$p_{ij}(m) \geq \prod_{a=1}^{n} p_{jj}(m_a)^{q_a} > 0.$$

Hence $p_{ij}(m) > 0$ for $m \geq M$.

Now since $i \leftrightarrow j$ choose $k$ so that $p_{ij}(k) > 0$.

Then $p_{ij}(m+k) \geq p_{ij}(k) p_{jj}(m) > 0$ if $m \geq M$.

Back to coupling

Let $\{X_n\} + \{Y_n\}$ be iid MC's with transition matrix $P$ which are irreducible + aperiodic. Set

$Z_n = (X_n, Y_n)$

Let $i, j, k, l \in S$. Let $N(i,j) \equiv N(k,l)$ be at

$P_{ij}(m) > 0$, $\forall m \geq N(i,j)$

$P_{kl}(m) > 0$, $\forall m \geq N(k,l)$.
Note that both \( N(i,j) \) and \( N(k,l) \) exist from the previous problem.

We now have \( \forall m \geq \max \{ N(i,j), N(k,l) \} \)

\[
P(\Xi_m = (k,l) | \Xi_0 = (i,j)) = p_{k,l}(m) \neq 0
\]

so that \( \{ \Xi_m \} \) is irreducible and aperiodic.

Remark. Take \( S = \{1,2,3\} \) and \( P = (0,1) \).

Then \( \{ (1,1), (2,2) \} \) and \( \{ (1,2), (2,1) \} \)

are closed sets of states for \( \{ \Xi_m \} \).

Obviously \( \{ \Xi_m \} \) is reducible. The reason

this happens is that \( \{ X_m \} \) is periodic (it is irreducible).

Once we have \( \{ \Xi_m \} \) irreducible and aperiodic we follow the coupling proof to get (\( \star \)) which then leads to

\[
\lim_{m \to \infty} P_{i,j}(m) = 0 = \frac{1}{E(T_{ij})}
\]
Sums of rv's

Let $X_0, X_1, \ldots$ be rv's, and set

$$S_t = X_0 + X_1 + \cdots + X_t$$

The process $\{S_t, t=0,1,\ldots\}$ is called a random walk. Typically, $X_1, X_2, \ldots$ are iid while $S_0 = X_0$ is treated as a constant $i$ and is the starting point of the walk. In the independent case $\{S_t\}$ is a Markov process if we add the iid assumption it is time homogeneous.

The simple random walk on $\mathbb{Z}$

Here, $X_1, X_2, \ldots$ are iid taking values $\pm 1$ with probabilities $p = P(X_i = 1) + q = P(X_i = -1)$ with $0 < p < 1$. Set the starting point $S_0 = i$. A typical sample path would look like

```
\begin{tikzpicture}
\draw[->] (0,0) -- (6,0) node[below] {time};
\draw (0,0) -- (0,1) node[left] {$i$} -- (1,0) -- (2,1) -- (3,1) -- (4,0) -- (5,1) -- (6,0);
\end{tikzpicture}
```
We have seen that this Markov Chain is irreducible and is recurrent in the symmetric case \( p = q = \frac{1}{2} \), however the process is null recurrent in the sense that average return times are infinite. When \( p \neq q \), then the process is transient. Set
\[
Y = \sum_{t=1}^{\infty} I_{A_t},
\]
where \( A_t = \{ S_t = i \} \). In the recurrent case \( Y \equiv +\infty \). In the transient case \( Y \) is geometric with \( E(Y) < \infty \) (the \( i \) is to remind us that we are conditioning on the initial state. We will drop it when this point is clear). So we have
\[
P(Y = \infty) = 1 \iff E(Y) = \sum P(A_t) = \infty \quad \iff \quad p = q = \frac{1}{2}
\]
Notice \( \{ Y = \infty \} = \{ A_t \ \text{i.o.} \} \) so that we have
\[
P(A_t \ \text{i.o.}) = 1 \iff \sum P(A_t) = \infty
\]
This reminds us of the Borel Cantelli Lemma and 0-1 Law. Certainly \( \sum P(A_t) < \infty \) \( \Rightarrow P(A_t \ \text{i.o.}) = 0 \) but not the \( A_t \) are not typically independent. It is tempting to approach the problem via tail events, but is \( \{ A_t \ \text{i.o.} \} \) one of them?
For random walks with general iid steps $X_k$, the recurrence problem is more complicated (see Breiman's text Probability, section 3.7). We will only consider integer steps $X_k$. Without loss of generality we take $S_0 = X_0 = 0$ so that we are then interested in the event $\{S_m = 0 \, \text{ for } \, m \geq 0\}$. We have

**Theorem** $E(X_k) = 0 \implies P(S_m = 0 \, \text{ for } \, m \geq 0) = 1$

**Proof** For any state $j$, let $T \geq 1$ be the first time $S_m = j$, and set

$$N_j = \sum_{m=1}^{\infty} I \{S_m = j\}$$

Then

$$E(N_j) = E[E(N_j \mid T)] = \sum_{t=1}^{\infty} \left[ \sum_{m=0}^{\infty} P(S_m = j \mid T = t) \right] P(T = t)$$

(since $m < T \implies S_m \neq j$)

$$= \sum_{t=1}^{\infty} P(T = t) \left[ 1 + \sum_{m=0}^{\infty} P(S_m = j \mid T = t) \right]$$
\[
= \sum_{t=1}^{\infty} P(T=t) \left[ 1 + E(N_0) \right]
\]
\[
= P(T<\infty) \left[ 1 + E(N_0) \right]
\]
\[
\leq 1 + E(N_0)
\]

Now take \( \epsilon > 0 \). For integer \( M > 0 \) we have
\[
\sum_{m=1}^{\infty} P(1S_mnow take \( m_0 \) large enough so that \( m \geq m_0 \) implies
\[
P(1S_m \leq \epsilon) \geq \frac{1}{2} \quad \text{say}.
\]
In particular, for $M$ large enough we have:

$$P(1S_m \leq M) \geq \frac{1}{2} \quad \text{for } m_0 \leq M \leq \frac{M}{e}$$

so that

$$1 + E(N_0) \geq \frac{1}{2M+1} \sum_{m=1}^{\infty} P(1S_m \leq M)$$

$$\geq \frac{1}{2M+1} \sum_{m_0 \leq m \leq \frac{M}{e}} P(1S_m \leq M)$$

$$\geq \frac{1}{2M+1} \sum_{m_0 \leq m \leq \frac{M}{e}} \frac{1}{2}$$

$$\geq \frac{1}{(2M+1)^2} \left( \frac{M}{e} - m_0 \right)$$

$$\to \frac{1}{4e} \quad \text{as } M \to \infty$$

It then follows (since $e > 0$ is arbitrary) that $E(N_0) = \infty$.

qed
The Ballot Problem ($\approx 1850$)

2 candidates in an election $A + B$.
Suppose $A$ wins. Let $\alpha = \#$ of votes for $A$ and $\beta = \#$ for $B$.
$P(A$ was always ahead $B)$

Assign $+1$ to a vote for $A$ and $-1$ to a vote for $B$.
As the votes come in you have
$0, X_1, X_2, \ldots, X_n$ ($n = \alpha + \beta$)

Let $S_t = 0 + X_1 + \cdots + X_t$

$\{ S_t \}$ is a random walk on the integers which starts at $0$. 

\begin{align*}
S_t
\end{align*}
The solution to the problem is to calculate \( P(S_t > 0 \text{ for } 0 < t \leq m) \) when starting from 0. In order to solve this consider our simple random walk \( \{S_t\} \) starting from a say (i.e. \( S_0 = a \)). Suppose we wish to calculate \( \sum P(S_m = b) \). Consider

\[
a \rightarrow a_1 \rightarrow a_2 \rightarrow \cdots \rightarrow a_m = b
\]

\( n \) arrows

Let \( r = \# \) of steps to the right (i.e. \# of +1's)
\( l = \# \) of steps to the left

Then \( r + l = m \) and \( r - l = b - a \) if we are to end up at \( b \) at time \( m \). Hence

\[
r = \frac{m + b - a}{2} \quad , \quad l = \frac{m - b + a}{2}
\]

so that

\[
P(S_m = b) = \binom{m}{r} \left( \frac{m + b - a}{2} \right) \left( \frac{m - b + a}{2} \right)
\]

\( \# \) of paths from \( a \) to \( b \)

req'd \( \rightarrow ( \text{with } \frac{m + b - a}{2} \text{ steps to the right}) \)
Suppose now that both \( a, b > 0 \). Set

\[
N_m(a,b) = \# \text{ of paths from } a \text{ to } b
\]

\[
N^0_m(a,b) = \# \text{ of paths from } a \text{ to } b \text{ which hit the time axis at least once}
\]

Every path from \( a \) to \( b \) which crosses the \( t \)-axis at least once has a reflection about the \( t \)-axis which is a path from \( -a \) to \( b \). The reverse is also true, so that

\[
N^0_m(a,b) = N^0_m(-a,b)
\]

We have just employed the reflection principle.

We are now in a position to solve the ballot problem. So, let \( b > 0 + a = 0 \). The number of paths from \( a=0 \) to \( b \) which do not revisit the \( t \)-axis is

\[
\frac{b}{m} N_m(0,b)
\]

To see this note that every such path first goes to 0 (more specifically \((1,1))\). Hence the number of desired paths is

\[
N_{m-1}(1,b) - N^0_{m-1}(1,b) = N_{m-1}(1,b) - N_{m-1}(-1,b)
\]

Now, \( N_m(a,b) = \left( \frac{m+b-a}{2} \right) \) so that the req'd \( \theta \) is
\[
\left( \frac{m}{m+1+b-1} \right) - \left( \frac{m}{m+1+b+1} \right) \\
= \left( \frac{m-1}{m+b} \right) - \left( \frac{m}{m+b} \right) = \left( \frac{m-1}{k-1} \right) - \left( \frac{m}{k} \right), \quad k = \frac{m+b}{2} \\
= \frac{(m-1)!}{(k-1)!(m-k)!} - \frac{(m-1)!}{k!(m-k-1)} \\
= \frac{k!(m-1)!}{k!(k-1)!(m-k)!} - \frac{m(m-1)!}{m!k!(m-k-1)(m-k)} \\
= \frac{k}{m} \frac{m!}{k!(m-k)!} - \frac{(m-k)}{m} \frac{m!}{k!(m-k)!} \\
= \frac{(m)}{k} \frac{k-(m-k)}{m} = \frac{2(k-m)}{m} \left( \frac{m}{k} \right) \\
= \frac{m+b-m}{m} \left( \frac{m}{m+b} \right) = \frac{b}{m} \quad N_m (0, b) \\
\]

For the ballot problem, \( m = a + p \), \( b = a - p \) (\( a = 0 \)) so that \( P(A \text{ is always ahead}) = \frac{b}{m} \frac{N_m (0, b)}{N_m (0, b)} = \frac{b}{m} \)

\[
= \frac{a-p}{a+p} \]
The Gambler's Ruin Problem

We consider a simple random walk $\{S_t\}$ starting at $j$, where $0 < j < a$. The process stops if either $S_t = 0$ or $S_t = a$. This Markov chain has $a + 1$ states with both 0 and $a$ being absorbing states (they are of course recurrent) and $\{1, \ldots, a-1\}$ being a transient class. There are 3 classes to this chain $\{0\}$, $\{1, \ldots, a-1\}$, $\{a\}$. The only nonzero transition probabilities (1-step) are

$$P_{i,i+1} = p \quad P_{i,i-1} = q \quad 0 < i < a$$

Set $\rho_j(t) = P(S_t = 0 | S_0 = j)$. This is also denoted by $\rho_j^n(t)$. We have (recall the backward equations) for $t > 0$

$$\rho_j(t) = p \rho_{j+1}(t-1) + q \rho_{j-1}(t-1) \quad 0 < j < a$$

$$\rho_0(t) = \rho_0(t-1), \quad \rho_a(t) = \rho_a(t-1)$$

with boundary conditions

$$\rho_j(0) = 1 \quad j = 0$$

$$\rho_j(t) = 0 \quad j > 0$$
"Clearly" \( p_i(t) \) increases with \( t \) and since it is bounded we have a limit \( p_i \) called the probability of ultimate ruin. If we let \( t \to \infty \) in the above equations we obtain

\[
p_j = p \, p_{j+1} + q \, p_{j-1} \quad (0 < j < a)
\]

with boundary conditions \( p_0 = 1 \), \( p_a = 0 \). This difference equation is easily solved yielding

\[
p_j = \frac{(q/p)^j - (q/p)^a}{1 - (q/p)^a}
\]

\[
= \frac{a - j}{a} \quad \text{if} \quad p = q = \frac{1}{2}
\]

In the extreme case \( a = \infty \) (playing against an infinitely rich opponent!) this reduces to

\[
p_j = (q/p)^j \quad p > q
\]

\[
= 1 \quad p \leq q
\]

These results can also be obtained via martingale methods (later).