Markov Chains

Consider \( \{ X_t, t \in T \} \) where \( T \subset \mathbb{R} \).

In fact \( T \) will be a set of time points. There are two cases of interest: \( T \).

\( T = \{ 0, 1, 2, \ldots \} \) \( \text{or} \) \( T = \{ \frac{t}{1+t \geq 0} \} \) \( \text{a discrete case} \)

\( T \) is "discrete" \& \( T \) is "cts". For example, \( T = \{ 0, 1, 2, \ldots \} \) \( \text{a cts case} \)

Consider \( U \) \{ possible values of \( X_t \}. \) This

is called the state space. The

Markov property states that \( \forall \)

\( t_1 < t_2 < \ldots \)

\( X_{t_{m+1}} \mid X_{t_m}, \ldots, X_{t_1} \overset{d}{=} X_{t_{m+1}} \mid X_{t_m} \)

\( \uparrow \) \( \uparrow \) \( \text{future } \) \( \text{present} \)

\( \Leftrightarrow E[g(X_{t_{m+1}}, X_{t_{m+2}}, \ldots) \mid X_{t_{m}}, \ldots, X_{t_1}] \)

\( = E[g(X_{t_{m+1}}, \ldots) \mid X_{t_m}] \), \( g \)
The Chapman-Kolmogorov Eq'n (CKE)

**Theorem.** If \( \{X_t\} \) is Markov, then for \( \forall \; t_1 < t_2 < t_3 \),

\[
E[h(X_{t_3}) | X_{t_1}] = E\{ E[h(X_{t_3}) | X_{t_2}, X_{t_1}] | X_{t_1} \},
\]

**Proof.** We have

\[
E[h(X_{t_3}) | X_{t_1}] = E\{ E[h(X_{t_3}) | X_{t_2}, X_{t_1}] | X_{t_1} \} = E\{ E[h(X_{t_3}) | X_{t_2}] | X_{t_1} \},
\]

the Markov property.

Transition probabilities for Markov Chains

\[
P_{ij}(t_1, t_2) = P(X_{t_2} = j | X_{t_1} = i)
\]
Note: For Markov Chains, the state space is countable, and we can label the states $1, 2, \ldots$

Set

$$p_{ij}(u) = p_{ij}(t, t+u)$$

If this does not depend on $t$ then we are in the time homogeneous case. This conditional probability is called a "$u$-step" transition probability. We can also form the matrix of these probabilities

$$\mathbf{P} = \sum p_{ij}(1) \mathbf{I}$$

$$P(k) = \sum p_{ij}(k) \mathbf{I}$$

$$R(t) = \begin{pmatrix} \frac{P(X_t = 1)}{P(X_t = 2)} \end{pmatrix}$$

$R(0)$ is the pdf of the initial distribution.
Clearly the CKE in the discrete time case \( T = \{0, 1, 2, \ldots, 3\} \) are

\[
P(m+m) = P(m) P(m)
\]

\[
\Rightarrow P(m) = P^m
\]

Also

\[
R(m') = R(m-1)' P
\]

\[
\Rightarrow R(m') = R(0)' P^m
\]

\[
\lim_{m \to \infty} R(m) ?
\]

If this exists, call it \( \Pi \) say, then

\[
\Pi' = \Pi' P
\]

so that \( \Pi \) is an eigenvector.
Recall
- state $j$ is accessible from state $i$
  if $p_{ij}(m) > 0$ for some $n > 0$.
  Notation $i \rightarrow j$
- $i \leftrightarrow j$ communicate if $i \rightarrow j$ and $j \rightarrow i$.
  Notation $i \leftrightarrow j$

Proposition $i \leftrightarrow i$, $i \rightarrow j \Rightarrow j \leftrightarrow i$,
  $i \leftrightarrow k + \times \leftrightarrow j$, then $i \leftrightarrow j$.

Proof. Obvious
$\leftrightarrow$ partitions the state space
State space

Within each member of the partition, all states communicate.

All states communicate $\Leftrightarrow$ MC is irreducible
i is recurrent if starting from i you are certain to return. Let \( T_i \) = time it takes to return. If \( E(T_i) < \infty \) then we call the state **positive recurrent**. A recurrent state with \( E(T_i) = \infty \) is **null recurrent**.

If not recurrent then **transient**.

The period of a state \( i \) is \( \gcd\{m : \pi_i(m) > 0\} \). Denote this by \( d(i) \). If \( d(i) = 1 \) then we call \( i \) **aperiodic**. Otherwise it is **periodic**.

Let \( T_{ij} \) = 1st time one enters state \( j \) starting from \( i \) (1st passage time).

\( T_{ii} \) is our \( T_i \). Set \( f_{ij}(m) = P(T_{ij} = m) \) to be the "PF" of \( T_{ij} \). Note \( \sum_m f_{ij}(m) \) may be \(< 1 \) so we may have an **improper distribution**.
Remark: \( T_{ij} \) may be \( +\infty \) with \( >0 \) prob.

For recurrent states \( T_{ii} \leq \infty \). For transient states \( P(T_{ii} = \infty) > 0 \).

\( \Rightarrow \) call \( f_{ij}(0) = 0 \), \( \forall i \neq j \).

Look at:

\[
P(T_{ij} < \infty) = \sum_{m=1}^{\infty} f_{ij}(m)
\]

Call the generating function.

\( \sum \pi_{ij}(m) : m=0,1,\ldots \) \( + \sum \phi_{ij}(m) : m=0,1,\ldots \)

\( P_i(s) + F_i(s) \) respectively. So

\[
P_i(s) = \sum_m \pi_{ij}(m) s^m, \quad F_i(s) = \sum_m \phi_{ij}(m) s^m
\]

Note when \( T_{ij} \leq \infty \) then \( F_{ij}(s) = E(s^{T_{ij}}) \)
Theorem \[ P_{i,j}(s) = s_{i,j} + F_{i,j}(s) \frac{P_{i,j}(s)}{s_{i,j}} \]

\([s_{i,j} = 1 \text{ if } i = j \quad 0 \text{ if } i \neq j]\)

Proof Suppose \( X_0 = i \). Then

\[
\sum_{m=0}^{\infty} I_{\{X_m = i\}} s^m = s_{i,j} + s_{i,j} \sum_{m=0}^{\infty} I_{\{X_m = i\}} e^{T_{i,j}}
\]

Now take \( E \)'s to get

\[ P_{i,j}(s) = s_{i,j} + F_{i,j}(s) \frac{P_{i,j}(s)}{s_{i,j}} \]

Notation \( P_{i,j}(s) = \lim_{s \to 0} P_{i,j}(s) \)

Theorem Statement is recurrent \( \iff P_{i,j}(1) = 0 \)

Proof \( i \) is recurrent \( \iff F_{i,j}(1) = 1 \)

Since

\[ F_{i,j}(s) = \frac{P_{i,j}(s) - 1}{P_{i,j}(s)} \]

we get \( P_{i,j}(s) \to 0^+ \text{ as } s \to 1^+ \). That is \( P_{i,j}(1) = 0 \). \( \qed \)