Background

\[ \mathbb{N} = \{1, 2, 3, \ldots \} \quad \text{natural #}'s \]
\[ \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots \} \quad \text{integers} \]

Sets which can be put into a 1-1 correspondence with \( \mathbb{N} \) are countably infinite. Sets which have a finite # of elements are countably finite. These two are the countable sets. Usually countable will mean countably infinite.

\( \mathbb{Z} \) is countable. So is \( \mathbb{Z}^2 \) so is \( \mathbb{Q} = \text{set of rationals} \)

\[ \frac{m}{n} \quad \text{integer} \]
\[ n \neq 0 \]

If a set is not countable it is uncountable. An example would be \([0, 1]\) or \( \mathbb{R} \).
Let $a_1, a_2, \ldots$ be a sequence. Consider
\[ \sup_{m \geq N^2} \{ a_m : m \geq N^2 \} = \inf_{m \geq N^2} \{ a_m : m \geq N^2 \} \]
(this is either finite or $\pm \infty$).

Now define
\[ \limsup_{m \to \infty} a_m = \lim_{N \to \infty} \sup_{m \geq N^2} \{ a_m : m \geq N^2 \} \]
This is either finite or $\pm \infty$. We will denote it by $\lim a_m$. In the same way,
\[ \liminf_{m \to \infty} a_m = \lim_{N \to \infty} \inf_{m \geq N^2} \{ a_m : m \geq N^2 \} \]
This is either finite or $\pm \infty$.

\[ \lim a_m = a \iff \limsup a_m = \liminf a_m = a \]

Also it is always true $\lim a_m \leq \lim a_m$.

This can be extended to $a(t)$, $-\infty < t < \infty$ in the obvious way so we have
\[ \limsup_{t \to \infty} a(t) \]
\[ a_n \to a \quad \text{if} \quad \forall \epsilon > 0 \quad \exists N \text{ such that } \quad m > N \Rightarrow |a_m - a| < \epsilon \]

\[ a_m \to a \Rightarrow a_{m_k} \to a \quad \text{as} \quad k \to \infty \]

where \( \{a_{m_k}\} \) is a subsequence.

By convention \( m_1 < m_2 < \ldots \)

On the other hand, if every \( a_{m_k} \to a \Rightarrow a_m \to a \)

Prop Let \( \{a_n\} \) be a sequence of \( \mathbb{R} \).

If every subsequence \( \{a_{m_{k_j}}\} \) has a further subsequence \( a_{m_{k_{j_k}}} \to a \)

then \( a_m \to a. \)
\[ a^n \rightarrow a \quad \text{if for every rational } \frac{1}{n} > 0 \exists \quad N \quad \text{and } m \geq N \]
\[ \Rightarrow |a_m - a| \leq \frac{1}{n} \]

\[ a^n \uparrow a \quad \text{if } a_1 \leq a_2 \leq \ldots \quad \text{and } a^n \rightarrow a \]

\[ a^n \downarrow a \quad \text{if } a_1 \geq a_2 \geq \ldots \]

\textbf{Fact:} \quad \lim_{x \to a} g(x) = l \quad \text{if } \forall x_m \rightarrow a

\[ g(x_m) \rightarrow l \quad \text{In fact this is } \quad g(x_m) \rightarrow l. \]

Another way of approaching \( a^n \rightarrow a \).

\[ a_m \rightarrow a \quad \text{if for every } \varepsilon > 0 \text{ only } \]
\[ a_m \rightarrow a \quad \text{if for every } \varepsilon > 0 \text{ only a finite number of the statements } |a_m - a| > \varepsilon \text{ hold.} \]
\((a, b)\) is an open interval
\(\{x \mid |x - a| < \varepsilon\}\) is an open disc (the inside of a circle/sphere of radius \(\varepsilon\)). A set \(B \subset \mathbb{R}^k\) is open if for every \(a \in B\) there is a disc \(\{x \mid |x - a| < \varepsilon\} \subset B\).

\([a, b]\) is clearly not open.

\[\begin{array}{c}
\boxed{a} \\
\text{disc is not a subset of } [a, b] \\
\boxed{b}
\end{array}\]

Suppose \(B \subset \mathbb{R}\) is open. Then \(B\) is a countable union of disjoint open intervals.

The rationals are dense in \(\mathbb{R}\).
Let \( G = \text{set of subsets of } V. \)
Then \( f^{-1}(G) = \{f^{-1}(B) \mid B \in G\} \)

Let \( S = \{w\} \) be a finite set.
The collection of all subsets of \( S \) is called the power set on \( S \). This power set includes \( S \) and \( \emptyset \). The number of elements in this power set is \( 2^{|S|} = 2^1 = 2^1 \).
Now suppose \( \mathbb{Z} = \mathbb{N} \). Would the power set be countable? No! The number of sequences of the type 1, 1, 0, 1, 0, \ldots is uncountable.

**Def'n** \( \mathcal{F} \) is a \( \sigma \)-field of subsets of \( \mathbb{N} \) if:

(i) \( \emptyset \in \mathcal{F} \)

(ii) \( A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F} \)

(iii) \( A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \bigcup_i A_i \in \mathcal{F} \)

Recall \( (\bigcup_t A_t)^c = \bigcap_t A_t^c \) \( \{ \text{de Morgan} \} \)

\( (\bigcap_t A_t)^c = \bigcup_t A_t^c \)
Each if $\{ F_t, t \in T \}$ is a collection of $\sigma$-fields then $\bigcap_{t \in T} F_t$ is a $\sigma$-field.

Let $C$ be a collection of subsets of $\mathbb{R}$. Clearly $C \subseteq \text{power set}$ is a $\sigma$-field if $F \in C$ is the $\sigma$-field generated by $C$ and is denoted by $\sigma(C)$. It is the smallest $\sigma$-field which includes $C$. 
On $\mathbb{R}^n$, the $\sigma$-field generated by the open sets is called the Borel $\sigma$-field and denoted by $\mathcal{B}_n$. This $\sigma$-field is very rich and includes open, closed sets etc.

Now consider

$$X : \Omega \rightarrow \mathbb{R}$$

Let $F_n$ be a $\sigma$-field on $\Omega$. If $X^{-1}(\mathcal{B}) \subseteq F_n$, then we say that $X$ is measurable with respect to $F_n$ and $\mathcal{B}$.

$(\Omega, F, P)$ is called a sample space probability space.
$X_n \xrightarrow{a.s.} X$  \hspace{1cm} $X_m \xrightarrow{p} X$

"with prob 1" \hspace{1cm} "almost surely"

If \( P(\lim_{n \to \infty} X_n = X) = 1 \). This is $\iff$

\[
P(\{\omega | X_m(\omega) \to X(\omega)\}) = 0
\]

Suppose for each rational $\epsilon_n > 0$

\[
P(\{1X_n - X > \epsilon_n, i.o\}) = 0
\]

\[
A_{\epsilon_n} \{1X_n - X > \epsilon_n, i.o\} \quad \text{is the event that an infinite number of events occur infinitely often.}
\]

Since \( P(U_{\epsilon_n} A_{\epsilon_n}) \leq \sum_{\epsilon_n > 0} P(A_{\epsilon_n}) = 0 \) we conclude that there is an event $A^c$ with $P(A^c) = 1$ such that for each $\epsilon > 0$

\[
P(\{\omega \in A^c | a \text{ finite } \# \text{ of } |X_n(\omega) - X(\omega)| > \epsilon\}) = 1
\]

Note $A^c$ does not depend on $\epsilon_n$. It then follows

$X_m \xrightarrow{a.s.} X$
More on convergence

Events $A_1, A_2, \ldots$ ($\infty$ #)

**Borel Cantelli Lemma**

(i) $P(A_{\infty 0}) = 0 \quad \text{if} \quad \sum P(A_k) < \infty$

(ii) $A_\text{ind} \quad \text{and} \quad \sum P(A_k) = \infty \quad \text{then} \quad P(A_{\infty 0}) = 1$

\[ X_n \overset{a.s.} \to X \quad (X_n \overset{w.p.} \to X) \]

Suppose
\[ \sum P(|X_n - X| > \epsilon) < \infty \]

**BCCL**
\[ \Rightarrow P(1_{|X_n - X| > \epsilon_0} \text{i.o}) = 0 \]

\[ \Rightarrow \text{All } X_n \text{ are within } \epsilon_0 \text{ of } X \text{ w.p.1} \]

Now let $B_n$ be the event $\text{st } P(B_n) = 1$. Set $B = \bigcap \cap B_n \Rightarrow P(B) = 1$. On $B$ all $X_n$ are within $\epsilon$ of $X$ for any $\epsilon > 0$.

\[ \Rightarrow X_n \overset{a.s.} \to X \]

**Lemma** Let $B_n$ be st $P(B_n) = 1$, $\forall n \in \mathbb{Q}$ then $P(\bigcap \cap B_n) = 1$
Proof
\[ P((\bigcap_n B_n)^c) \]
\[ = P(\bigcup_n B_n^c) \leq \sum_n P(B_n^c) = 0 \]
\[ \Rightarrow P((\bigcap_n B_n)^c) = 0 \quad \Rightarrow P(\bigcap_n B_n) = 1 \] \( q.e.d. \)

\[ X_m \rightarrow X \quad (P(|X_m - X| > \epsilon) \rightarrow 0) \]
\[ \Rightarrow \exists X_{m_k} \xrightarrow{a.d.} X \]

**Example** Suppose \( X_m \rightarrow X \) \& \( |X_m| \leq W \) where \( E(W) < a \). Then \( E(X_m) \rightarrow E(X) \).

Proof Set \( a_m = E(X_m) \) \& \( a = E(X) \). Let \( a_{m_k} \) be any subsequence of \( \{a_m\} \).

Now look at \( X_{m_k} \rightarrow X \). Then there is a subsequence of this subsequence, say \( X_{n_k} \) such that \( X_{n_k} \rightarrow X \). Now use the
DCT to get
\[ E(x_{m_n}) \to E(x) \]

That is
\[ a_{m_n} \to a \]

so
\[ a_m \to a \]

A bit more on \( \sigma \)-fields

\( (a, b) \leftarrow \) open interval \( \subset \mathbb{R} \)

open subset of \( \mathbb{R} \)

closed set = complement of an open set

open set on \( \mathbb{R} = \) countable union of disjoint open intervals

open set on \( \mathbb{R}^n = \) countable union of open "intervals"

\[ \{ \exists \alpha < x < \beta \} \]

On \( \mathbb{R} \)
\( \mathcal{B} = \sigma(\text{open sets}) = \) Borel \( \sigma \)-field
\[ = \sigma(\text{open intervals}) = \sigma(\text{closed sets}) \]
\[ \sigma(\{(a, b] \}) = \sigma(\{(\infty, b] \}) \text{ etc...} \]

Now let \( g : \mathbb{R} \to \mathbb{R} \) be cts.
Then \( g^{-1}(\text{open set}) \) is open which means \( g^{-1}(B) \subset B \Rightarrow g \) is measurable w.r.t. \( B \) or \( \mathcal{B} \) (also true \( g : \mathbb{R}^m \to \mathbb{R}^m \)).

Look at all the cts f's \( f : \mathbb{R} \to \mathbb{R} \) and limits of convergent sequences of cts f's. These are the \textit{Baire f's} (\& they = set of \textit{measurable} f's \text{ wrt } \mathcal{B})

(Also true from \( \mathbb{R}^m \to \mathbb{R} \))

\text{Proposition}: Let \( X : L^2 \to \mathbb{R}^m \) be a r vec \& \( g : \mathbb{R}^m \to \mathbb{R} \) be cts.
Then \( g(X) : L^2 \to \mathbb{R} \) is \textit{measurable rv}.
Take a σ-field $F$. Suppose for the moment that $F$ is countably the nonempty elements of $F$ are $A_1, A_2, \ldots$

Let $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, \ldots\}$. $\mathcal{E}$ is uncountable.

Set $A = A_1 \cup A_2 \cup \cdots$

$A_m = \bigcup_{\mathcal{E}_i \subseteq \mathcal{E}_m} A_i$

$A_m^\circ = A_m^c$

$A_m^1 = A_m^\cap$

$\Rightarrow$ countable $\mathcal{M}$ of $A_i$ disjoint $\mathcal{M} \cap F = 0$

$\Rightarrow F$ is uncountable $\in F$
A bit more on rv's & other matters

\((\Omega, \mathcal{F}, P)\) - probability space

\(\sigma\)-field

If \(A \in \mathcal{F}\) & \(P(A) = 0\) & \(A_0 \subseteq A\) then we will assume \(A_0 \in \mathcal{F}\). That is \(\mathcal{F}\) is complete.

Recall \(P\) satisfies

1. \(P(\Omega) = 1\) - finite
2. \(P(A) \geq 0\) - positive
3. \(P(\sum A_i) = \sum P(A_i)\) - \(\sigma\)-additive

Drop \(P\Rightarrow\) measure \(\mu(A)\). If \(P\)

can be partitioned into sets \(\Omega_1, \Omega_2, \ldots\)

such that \(\mu(\Omega_i) < \infty\) then \(\mu\) is a \(\sigma\)-finite measure. In this case

\(\mu = \sum P_i\)
Note \( M(A) = c_1 P_1(A) + c_2 P_2(A) + \cdots \). The \( P_i \) are restricted to \( \Omega_i \). In this case

\[
\int_X d\mu = c_1 \int_X dP_1 + c_2 \int_X dP_2 + \cdots
\]

\( E_1(X) \quad E_2(X) \)

The \( c_i \)'s are just the \( \mu(\Omega_i) \). The

DCT + MCT continue to hold for these integrals.

Bohr o-field

eg. Look at \( \mathbb{R} \) with \( \mathcal{B} \). Let \( \mathcal{B} \) be a Borel o-field.

Let \( P_i \) be uniform on \( (i, i+1) \).

Define the length of \( \mathcal{B} \) \( \mathcal{B} \) as

Lebesgue \( \lambda(B) = \sum_{i \in \mathbb{Z}} P_i(B) \)

measure
\[ \int f(x) \, dx \text{ is the Lebesgue integral. You see it as } \int g(x) \, dx. \]

Note \( g : \mathbb{R} \to \mathbb{R} \) is mbe wrt \( \mathcal{B} = \mathcal{C} \).

\[ X \xrightarrow{\text{O-field}} G \]

\[ X^{-1}(B) \subset G \]

\[ Y^{-1}(B) \subset X^{-1}(B) \]

If \( Y = h(X) \) then \( Y^{-1}(B) \subset X^{-1}(B) \)

Start with \( X \xrightarrow{\text{O-field}} X^{-1}(B) \)

O-fields which are subsets of \( X^{-1}(B) \) correspond to rv's which are f'm of \( X \).
We have defined $E(Y|X)$ as a function of $X$. It is convenient to define $E(Y|X)$ as that 0-field $rv$ which is $A \times B$ mle, which best predicts $Y$. Most of the time (for us) $A$ corresponds to a rv $X$ which is why we started with $E(Y|X)$.

Finally, for any $m, m' \in \mathbb{Z}$, $r \in [0, 1)$ such that
$$m = km + r$$

Define The gcd of integers $m$ and $m'$ is an integer $k > 0$ such that $k|m$ and $k|m'$ and is $l > 0$ divides $m$ and $m'$ then $l|m$. 
Remark
1. The gcd is just the largest divisor.
2. If \( i, j \geq 0 \) have \( \gcd = 1 \) then
   \[
   \left\{ c, i + c_2 j \mid c, c_2 \in \mathbb{Z}^+ \right\}
   \]
   includes all of \( \mathbb{Z}^+ \) except possibly for a finite set. Is this still true for \( i, \ldots, i_k \) with \( \gcd = 1 \)?
   Yes.

\[\text{EA} \text{ if } m \in \mathbb{Z}, m \in \mathbb{Z}^+ \Rightarrow q \in \mathbb{Z} + r \in \mathbb{Z}^+ \]
where \( 0 \leq r < m \) such that
\[
m = qm + r
\]

If \( i, j \) are such that \( \gcd = 1 \) then \( \exists \)
integers \( a \times b \) st \( ai + bj = 1 \)
\[
\Rightarrow \left\{ c, i + c_2 j \mid c, c_2 \in \mathbb{Z}^+ \right\} \text{ includes all of } \mathbb{Z}^+ \text{ except possibly a finite set}.
\]
Also true for \( i, \ldots, i_k \geq 0 \) with \( \gcd = 1 \)
\[
\exists \text{ integers } a, \ldots, a_k \text{ st } a, i_1 + \ldots + a_k i_k = 1
\]
\( \mathbb{R}, \mathcal{B} = \text{Borel } \sigma\text{-field} \)
\( \mathbb{R}^n, \mathcal{B}_n = \cdots \)
\( \mathbb{R}^\infty = \left\{ \left( \frac{x_i}{i} \right) \right\}, \mathbb{R}^m = \left\{ \left( \frac{x_i}{i} \right) \right\} \)
\( \mathcal{B}_\infty = \text{Borel } \sigma\text{-field on } \mathbb{R}^\infty \)

\((\mathbb{R}, \mathcal{F}, \mathbb{P})\)

\(X \sim (\mathcal{B}_m) \subset \mathbb{R} \)
\(\sigma\text{-field generated by } X - \sigma(X) \)

Let \( X \in \mathbb{R}^\infty \) and suppose we want to calculate \( \mathbb{P}(X \in B) \). This can be approximated by \( \mathbb{P}(X_m \in B_m) \).
Notice that for $B_1$ on $\mathbb{R}$, it is generated by sets of the type $B_1$. Thus the smallest $\sigma$-field including these types of sets is $B_1$. For $B_2$ on $\mathbb{R}^2$ we generate it by product sets of the type $B_1 \times B_2$.

So $B_2 = \sigma(\{B_1 \times B_2\})$.

For $\mathbb{R}^\infty = \{(x_i)\}$, $B_\infty = \sigma(\{B_1 \times B_2 \times \ldots\})$. 
If $X$ is a r vec ($f$'m from $\mathbb{R}^m$)
\[ \sim \]
then $\{ X \in B \}$ in an event in $\sim$. We then denote it by $X^{-1}(B)$. Formally, it is $\{ \omega : X(\omega) \in B \}$. The collection of such events, $\{ X^{-1}(B) : B \in \mathcal{B}_m \}$, is a $\sigma$-field in $\sim$ and is denoted by $\sigma(X)$ or $X^{-1}(B)$. Notice $\sim$ induces the new probability space $(\mathbb{R}^m, \mathcal{B}_m, P)$, where $\sim$ $\sim$

Now take $g : \mathbb{R}^m \to \mathbb{R}$ to be cts + hence measurable w.r.t. $\mathcal{B}_m + \mathcal{B}$. The composition $g \circ X = g(X)$ is of course measurable w.r.t. $\sim + \mathcal{B}$. This only requires $g$ to be mbl w.r.t. $\mathcal{B}_m + \mathcal{B}$ (not necessarily cts).
Take a countably ∞ # of events $A_1, A_2, \ldots$ We denote the event 
\{an \infty \# of the $A_i$'s occurs\} by $\{A_m \text{ i.o.}\}$ or just $A_m \text{ i.o.}$. It is also denoted by:

$$\lim \sup_{m=1}^\infty A_m = \bigcap_{m=1}^\infty \bigcup_{m=m}^\infty A_m$$

**Boole-Cantelli**

(a) $\sum_{m=1}^\infty P(A_m) < \infty \implies P(A_m \text{ i.o.}) = 0$

(b) $\sum_{m=1}^\infty P(A_m) = \infty \implies P(A_m \text{ i.o.}) = 1$

**Proof**

(a) $P(A_m \text{ i.o.}) = P\left(\bigcap_{m=m}^\infty \bigcup_{m=m}^\infty A_m\right)$

$$= P\left(\lim_{m \to \infty} \bigcup_{m=m}^\infty A_m\right)$$

$$= \lim_{m \to \infty} P\left(\bigcup_{m=m}^\infty A_m\right)$$

$$\leq \lim_{m \to \infty} \sum_{m=m}^\infty P(A_m) \quad \text{(Boole)}$$

$$= 0 \quad \text{(series converges)}$$
\[
\Pr(\{A_m \cap \cap c\}) = \Pr\left(\bigcap_{m=\infty}^{\infty} A_m^c\right)
\]

= \Pr(\lim_{m\to\infty} \bigcap_{m=\infty}^{\infty} A_m^c)

= \lim_{m\to\infty} \Pr(\bigcap_{m=\infty}^{\infty} A_m^c)

= \lim_{m\to\infty} \prod_{m=\infty}^{\infty} P(A_m^c)

= \lim_{m\to\infty} \prod_{m=\infty}^{\infty} [1 - P(A_m)]

\leq \lim_{m\to\infty} \prod_{m=\infty}^{\infty} e^{-P(A_m)}

= \lim_{m\to\infty} e^{-\sum_{m=\infty}^{\infty} P(A_m)} = 0

\quad \text{\textit{q.e.d.}}

Notice that the occurrence of \( \{ A_{n \in \mathbb{N}} \} \) does not depend on any finite # of the \( A_i \)'s. It is not a coincidence that \( P(A_{n \in \mathbb{N}}) \) is either 0 or 1 (in the independent case). This is an example of a \textbf{Zero-One Law}.

By the way, notice:

\[ \text{eg } X = c \implies X^{-1}(B) = \{ \emptyset, \{ c \} \} \]

This would also be the case if \( X = c \) (that is, \( X^{-1}(B) \) would consist of events having prob 0 or 1."

\[ \]
Look at a sequence of rv's
\[ X_1, X_2, \ldots, X_m, X_{m+1}, \ldots \in \mathcal{X} \]
Look at all \( \bigcap \sigma(X_m, X_{m+1}, \ldots) \)
\[ \subset \mathcal{F} \]
\[ \bigcap \sigma(X_m, X_{m+1}, \ldots) = \text{Tail } \sigma\text{-field} \]
a \( \sigma\)-field
denoted as \text{Tail events}

\text{eg. Let } X_1, X_2, \ldots \text{ be i.i.d with mean } \mu \text{ and set }
\[ \overline{X_m} = \frac{X_1 + \ldots + X_m}{m} \]

\text{let } A = \{ \overline{X_m} \text{ converges } \}
This is a tail event!