Let $X > 0$ be integer valued. We say it is aperiodic or non-arithmetic if it is not confined to the grid $\mathbb{Z}^d$ where $d > 1$. If it were then $\frac{x}{d}$ would be non-arithmetic. Another way of saying this is to require

$$\gcd \{ m \mid P(X=m) > 0 \} = 1$$

Suppose $\gcd \{ m \mid P(X=m) > 0 \} = \{ m_1, m_2, \ldots \}$.

**Lemma** $\gcd \{ m_1, m_2, \ldots \} = 1 \implies \exists \text{ a finite set } \{ m_1, \ldots, m_k \} \text{ with } \gcd = 1$.

**Proof** Let $N = \gcd \{ m_1, \ldots, m_k \}$. Then $N$ is decreasing (nonincreasing) and hence has a limit. The limit must be in $1/N$. Suppose it is $i$. Then $\forall l$ large enough $N = i$. Then $\forall l$ large enough $N = i$. Hence $i = 1$ and there is a finite collection of $m_i$s with $\gcd = 1$.
Another simple proof of this lemma starts with the smallest \( \neq 0 \) element of \( \{m_1, m_2, \ldots\} \). Factor it into primes \( p_1, \ldots, p_k \) say. Now select one of the \( m_i \)'s for which \( p_1 \) is not a factor. Select another for which \( p_2 \) isn't. This yields \( k+1 \neq \phi \) with \( \gcd \ 1 \).

**Lemma** Let \( m_1, \ldots, m_k \in \mathbb{N}^+ \) be such that \( \gcd \{m_1, \ldots, m_k\} = 1 \). Then \( \exists \) integers \( l_1, \ldots, l_k \) such that \( 1 = l_1 m_1 + \cdots + l_k m_k \).

**Proof** Let \( S = \{m'm | m \in \mathbb{Z}^+\} \) and set \( c = \min S \cap \mathbb{N}^+ \) (obviously \( S \cap \mathbb{N}^+ \) is not empty - take \( c = 1 \) for example). Then \( c = \pi_1(\ell')m' \). Then we have
\[
m'm = \pi_1(\ell'm') + r
\]
\[
\Rightarrow r = (m - \pi_1\ell')m' \in S.
\]
Hence $r = 0$ (since $r > 0$ and $r < c = \min \mathbb{S} \backslash \mathbb{N}$).

So $x = qc$ so that $c$ divides $x$ and each of $m_1, \ldots, m_k$. But $\gcd \mathbb{E} \mathbb{m}_1, \ldots, m_k = 1$ so that $c = 1$.

\[ \gcd \]

**Corollary** Let $m_1, \ldots, m_k \in \mathbb{N}$ have $\gcd = 1$. Then \[ \{ m \in \mathbb{Z}^k \mid \exists \, n \in \mathbb{Z} \} = \mathbb{Z} \]

**Remark** We are using the notation \[ m' = (m_1, \ldots, m_k) \] and \[ m'' = (m_1, \ldots, m_k) \] and so on. \( \mathbb{Z}^k = \{ m \} \). Had the gcd been denoted we would have \( \mathbb{Z}^k = \mathbb{N} \).
Let \( X \in \mathbb{Z}^+ \) be aperiodic (non-arithmetic) in the sense that \( X \) is not confined to a lattice \( \{md : m = 0, 1, \ldots \} \) for integer \( d > 1 \). For such a \( X \) \( \exists k_1, \ldots, k_\ell \) relatively prime (i.e. \( \text{gcd} = 1 \)) with \( P(X = k_i) > 0 \), \( i = 1, \ldots, \ell \).

It is then the case that \( \exists \ N > 0 \) st every \( m \geq N \) can be written as a positive \((\geq 0)\) linear combination of \( k_1, \ldots, k_\ell \) (cf. Feller, Vol I, Chap 13, section 11 - Lemma 1). Now let \( X_1, X_2, \ldots \) be iid \( X \). If \( m \geq N \) we then write \( m \) as \( m, k_1, \ldots, + m_\ell k_\ell \), where the \( m_i \)'s are \( \geq 0 \). Set \( M_0 = m_1 + \cdots + m_\ell \) \((\geq 1)\). We then have

\[
P\left( \sum_{i=1}^{M_0} X_i = m \right) \geq \prod_{i=1}^{\ell} \left[ P(X = k_i) \right]^{m_i} > 0
\]

Now, for any integer \( z \) we can find \( m, m \geq N \) s.t. \( z = m - m' \). Take \( X_1, X_2, \ldots \) to be iid \( X \) st each of the \( X_i \)'s. We then have \( N_0 \geq 1 \) s.t.

\[
P\left( \sum_{i=1}^{N_0} Y_i = m \right) > 0
\]

\[
P\left( \sum_{i=1}^{M_0} X_i - \sum_{i=1}^{N_0} Y_i = z \right) > 0 \quad (\text{x})
\]
Proposition Let \( X_1, X_2, \ldots, Y_1, Y_2, \ldots \) be \( > 0 \) non-arithmetic rv's which are i.i.d. and independent of some other rv \( Z \in \mathbb{Z} \) (integer). Then \( \exists \) rv's \( M, N > 0 \) such that

\[
    \sum_{i=1}^{M} X_i - \sum_{j=1}^{N} Y_j \equiv Z
\]

Proof: Condition on \( Z = z \). Since \( \sum_{i=1}^{\infty} (X_i - Y_i) \) is a \( 0 \)-mean random walk on the integers \( \exists N_1 < N_2 < \cdots \Rightarrow \sum_{i=1}^{N_k} (X_i - Y_i) = 0, \forall k \) wp 1.

Now let \( M_0, N_0 \) be as in our previous proposition. Set

\[
    \alpha = P\left( \sum_{i=1}^{N_k+M_0} X_i - \sum_{j=1}^{N_k+N_0} Y_j = z \right)
\]

Then \( \alpha > 0 \) by our previous proposition and the definition of \( N_k \). Notice \( \alpha \) is the same for each \( k \). Now take a subsequence \( \{N_{k_2}\} \) of \( \{N_k\} \) such that \( N_{k_2+1} - N_{k_2} > \max(M_0, N_0) \) and set \( A_\ell = \left\{ \sum_{i=1}^{N_{k_2}+M_0} X_i - \sum_{j=1}^{N_{k_2}+N_0} Y_j = z \right\}, \ell = 1, 2, \ldots \)
and
\[ A'_l = \left \{ \sum_{N_k' + 1}^{N_k' + M_0} X_i - \sum_{N_k' + 1}^{N_k' + N_0} Y_j \right \} = z_j^l , \quad l = 1, 2, \ldots . \]

We have \( P(A'_2) = \alpha \) and the \( A'_2 \) are independent. So \( P(A'_2 \cap 0) = 1 \) so that \( P(A'_2 \cap 0) = 1 \). Now uncondition the \( \mathbb{Z} \) to get
\[
P(A'_2 \cap 0) = \sum_\mathbb{Z} P[\{A'_2 \cap 0 \mid \mathbb{Z} = z\}] \cdot P(\mathbb{Z} = z)
= \sum_\mathbb{Z} P(\mathbb{Z} = z) = 1 \]

Since the first part of the proof assumed \( \mathbb{Z} = z \) and actually showed \( P \left( \mathbb{Z} = z \mid \mathbb{Z} = z \right) = 1 \).

\( \textit{qed} \)

Remark. This result shows that \( P(T < \infty) = 1 \) for the \( T \) in the coupling proof of the renewal theorem.