Problem Set #3 ex'd

(a) Let \( \{Y_x\} \) with \( Y_x \geq 0 \) be u.i. Show \( \sum_{x \geq 0} E(Y_x) < \infty \).

(b) Let \( \{X_x\} \) be u.i. Suppose \( X_x \geq 0 \). Show \( \sum_{x \geq 0} E(X_x) = E(X) + E(X_{X > 1}) \).

(c) Let \( X_m, X \rightarrow X \). Show \( \{X_m\} \) is u.i.

(d) Let \( X_m \in L_1 \), \( X_m \rightarrow X \) in \( L_1 \). Show \( \{X_m\} \) is u.i.

2(a) Let \( X_1, X_2, \ldots \) satisfy \( E(X_m | X_{m-1}) = 0 \), \( \forall m \geq 1 \), where \( X_k = (X_1, \ldots, X_k)' \) and \( X_0 = 0 \) for convenience.

Set \( S_m = X_1 + \cdots + X_m \). Show \( E(S_m | S_{m-1}) = S_m \).

For \( 1 \leq m < n \) show \( E(S_n) = \sum_{k=1}^{n} \text{Var}(X_k) \).

(b) For the situation in 2(a) show \( \text{Var}(S_m) = \sum_{k=1}^{m} \text{Var}(X_k) \).

(c) If \( \{X_m\} \) and \( \{S_m\} \) are as in 2(a) and \( \{Y_m\} \)

is such that \( X_m = g(Y_m) \) and \( E(S_m | X_m) = S_m \)

for \( 1 \leq m < n \) show \( E(S_n | S_{m-1}) = S_m \), \( 1 \leq m < n \)

while \( E(X_m | X_{m-1}) = E(X_m | Y_{m-1}) = 0 \), \( \forall m \geq 1 \)

(\( X_0 = Y_0 = 0 \)).

Remark: \( \{S_m\} \) in 2(a) is a (zero mean) martingale.

- \( \{S_m\} \) in 2(b) is a martingale wrt \( \{Y_m\} \).

- \( \{c + S_m\} \) is a martingale with \( E(c + S_m) = c \).
3(a) Let \( W_1, W_2, \ldots \) be a sequence. \( f(\cdot; \theta) \) is a function of \( \theta \). Show, assuming reasonable conditions, that
\[
\left\{ \frac{d \log L_n(\theta)}{d \theta} \right\}
\]
is a martingale. Here, \( L_n(\theta) = f(W_n; \theta) \) is the likelihood function. In this context, it is usual to avoid the upper/lowercase notation for r.v.'s.

(b) Let \( X_1, X_2, \ldots \) be iid with \( E(X_i) = 0 \), \( \forall i \). Show \( \{ S_n \} \) is a martingale.

(c) Let \( \{ Z_m, m \geq 0 \} \) be a branching process with \( Z_0 = 1 \), offspring mean \( \mu \) and probability of ultimate extinction \( p \). Show \( \{ Z_n / E(Z_n) \} \) and \( \{ p^Z_m \} \) are both martingales wrt \( \{ Z_m \} \). Now add immigration in each generation with mean \( m \). Show \( \frac{1}{m^m} \left[ Z_m - m \left( \frac{1 - \mu^m}{1 - \mu} \right) \right] \) is also a martingale (assume \( \mu \neq 1 \)).

(d) Let \( \{ Z_m \} \) be a Markov chain with state space \( \mathbb{R} \), satisfies \( P h = h \), where \( P \) is the transition matrix. Show \( \{ h(Z_m) \} \) is a martingale.