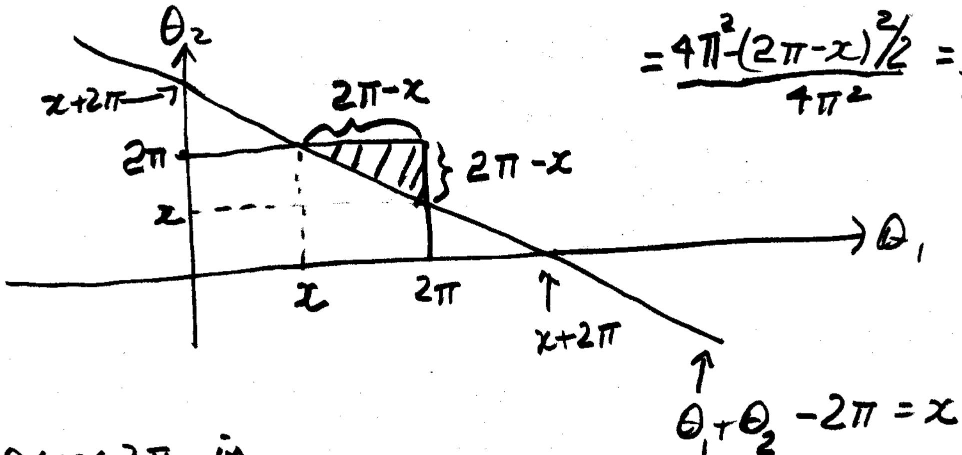
p21#5 In class we showed θ_1, θ_2 to be iid uniform$(0,2\pi)$
To get the pdf of X calculate its df & then differentiate.
So, if $0 < x < 2\pi$ then $P(X \le x) = [4\pi^2 - \text{shaded area}] \times \frac{1}{4\pi^2}$

$$= \frac{4\pi^2 - (2\pi - x)^2/2}{4\pi^2} = \frac{1}{4\pi^2}\left(\frac{4\pi^2 + 4\pi x - x^2}{2}\right)$$



The pdf for $0<x<2\pi$ is

$$\frac{2\pi - x}{4\pi^2}$$

& for $-2\pi < x < 0$ a similar argument yields $\frac{2\pi + x}{4\pi^2}$

so that the pdf is $\frac{2\pi - |x|}{4\pi^2}$, $-2\pi < x < 2\pi$

p32#7 On the basis of $Y_0, \ldots, Y_t$ we wish to predict Y_{t+s}
using an LLS estimate. It turns out this is

$$\hat{Y}_{t+s} = \mu + \beta^s (Y_t - \mu)$$

We need to calculate $V_{\underset{\sim}{Y}\underset{\sim}{Y}} = Var(\underset{\sim}{Y})$, where $\underset{\sim}{Y} = \begin{pmatrix} Y_0 \\ \vdots \\ Y_t \end{pmatrix}$

and $V_{\underset{\sim}{Y} Y_{t+s}} = \begin{pmatrix} cov(Y_0, Y_{t+s}) \\ \vdots \\ cov(Y_t, Y_{t+s}) \end{pmatrix}$

Now

$$V_{\underset{\sim}{Y}\, Y_{t+s}} = \begin{pmatrix} \alpha\beta^{t+s} \\ \vdots \\ \alpha\beta^{s} \end{pmatrix} = \alpha\beta^{s} \begin{pmatrix} \beta^{t} \\ \vdots \\ \beta^{0}=1 \end{pmatrix}$$

$$V_{\underset{\sim}{Y}\underset{\sim}{Y}} = \begin{pmatrix} \alpha & \alpha\beta & \cdots & \alpha\beta^{t} \\ \alpha\beta & \ddots & & \vdots \\ \vdots & & \ddots & \alpha\beta \\ \alpha\beta^{t} & \cdots & \alpha\beta & \alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 & \beta & \beta^{2} & \cdots & \beta^{t} \\ \beta & 1 & \ddots & & \beta^{2} \\ \beta^{2} & \ddots & \ddots & \ddots & \beta \\ \vdots & & & & \vdots \\ \beta^{t} & \cdots & \beta^{2} & \beta & 1 \end{pmatrix}$$

Set $\underset{\sim}{a} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \beta^{s} \end{pmatrix}$. Then the claim is $\hat{Y}_{t+s} - \mu = \underset{\sim}{a}'(\underset{\sim}{Y} - \mu \underset{\sim}{1})$

and this will hold if

$$V_{\underset{\sim}{Y}\underset{\sim}{Y}}\, \underset{\sim}{a} = V_{\underset{\sim}{Y}\, Y_{t+s}} \qquad (*)$$

That (*) holds is obvious (matrix multiplication) so we are done

Note

$$\begin{pmatrix} 1 & \beta & \beta^{2} \\ \beta & 1 & \beta \\ \beta^{2} & \beta & 1 \end{pmatrix}^{-1} = \frac{1}{1-\beta^{2}} \begin{pmatrix} 1 & -\beta & 0 \\ -\beta & 1+\beta^{2} & -\beta \\ 0 & -\beta & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \beta & \beta^{2} & \beta^{3} \\ \beta & 1 & \beta & \beta^{2} \\ \beta^{2} & \beta & 1 & \beta \\ \beta^{3} & \beta^{2} & \beta & 1 \end{pmatrix} = \frac{1}{1-\beta^{2}} \begin{pmatrix} 1 & -\beta & & 0 \\ -\beta & 1+\beta^{2} & -\beta & \\ & -\beta & 1+\beta^{2} & -\beta \\ 0 & & -\beta & 1 \end{pmatrix}$$

etc...

so that $V_{\underset{\sim}{Y}\underset{\sim}{Y}}^{-1}$ exists + the sol'n is unique.

p38 # 14

$$P(|\overline{Y}-\mu|>\epsilon) = P((\overline{Y}-\mu)^2>\epsilon^2)$$
$$\leq \frac{E[(\overline{Y}-\mu)^2]}{\epsilon^2} = \frac{Var(\overline{Y})}{\epsilon^2}, \text{ as } \mu = E(\overline{Y})$$
$$= \frac{\sigma^2}{n\epsilon^2} \to 0$$

Hence $\overline{Y} \xrightarrow{P} \mu$

Note This same appeal to Markov's Inequality shows that $X_n \xrightarrow{ms} X \Rightarrow X_n \xrightarrow{P} X$ holds in general.

p57 # 2

$$\binom{N}{r} p^r (1-p)^{N-r} = \frac{N!}{(N-r)!\,r!} p^r(1-p)^{N-r}$$
$$\sim \frac{\sqrt{2\pi}\, e^{-N} N^{N+\frac{1}{2}}}{r!\sqrt{2\pi}\, e^{-N+r}(N-r)^{N-r+\frac{1}{2}}} p^r(1-p)^{N-r}$$
$$= e^{-r}\left(\frac{N}{N-r}\right)^{N+\frac{1}{2}} \frac{(N-r)^r p^r (1-p)^{N-r}}{r!}$$
$$= e^{-r}\frac{1}{\left(1-\frac{r}{N}\right)^{N+\frac{1}{2}}} \frac{(pN-rp)^r}{r!}\left(1-\frac{\rho}{N}\right)^{N-r}$$
$$\to e^{-r}\frac{1}{e^{-r}}\frac{\rho^r e^{-\rho}}{r!} = e^{-\rho}\frac{\rho^r}{r!}$$

p60 #4

X & Y ind $\Rightarrow$ $\mathrm{cov}(X,Y) = E(XY) - E(X)E(Y)$

$$= E(X)E(Y) - E(X)E(Y) = 0$$

so that X & Y are uncorrelated.

p60 #6

Condition on $X_0 = x$. Then if T = time to the 1st time he beats X_0 (ie his 2nd record) then we have a geometric with

$$P(T=m \mid X_0 = x) = F(x)^{m-1}(1-F(x))$$

Hence

$$P(T=m) = \int_0^\infty P(T=m \mid X_0 = x) f(x)\,dx$$

$$= \int_0^\infty F(x)^{m-1}(1-F(x)) f(x)\,dx = \int_0^1 u^{m-1}(1-u)\,du,$$

where we have set $u = F(x)$. This is just $\frac{1}{m} - \frac{1}{m+1}$.

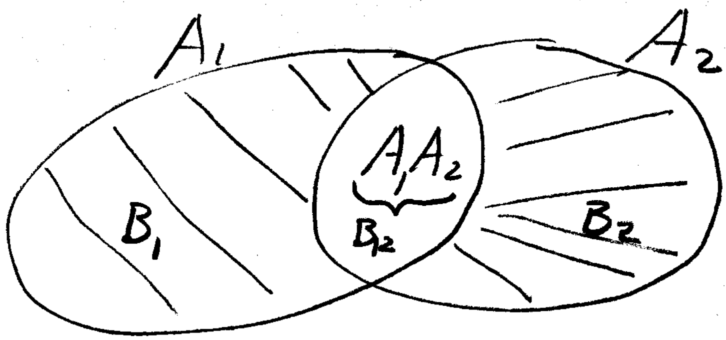
p65 #'s 1,2 done in STA 257 & #4 done in class

p65 #3 $E[(S_r - r)^{(k)}]$ is obtained by differentiating the pgf of $S_r - r$ k times. The pgf is

$$G(s) = E(s^{S_r - r}) = \frac{1}{s^r} E(s^{S_r}) = \frac{1}{s^r}\frac{(ps)^r}{(1-qs)^r}$$

$$= \frac{p^r}{(1-qs)^r}, \quad |s| < 1/q$$

p65 #5



We have

$$N(A_1) = N(B_1) + N(B_{12})$$
$$N(A_2) = N(B_2) + N(B_{12}) \,,$$

where $A_1 = B_1 \cup B_{12}$, $A_2 = B_2 \cup B_{12}$ & the B's are disjoint. The joint pgf of $N(A_1)$ & $N(A_2)$ is

$$G(s_1, s_2) = E\left[s_1^{N(A_1)} s_2^{N(A_2)}\right]$$
$$= E\left(s_1^{N(B_1)} s_2^{N(B_2)} (s_1 s_2)^{N(B_{12})}\right)$$
$$= E\left(s_1^{N(B_1)}\right) E\left(s_2^{N(B_2)}\right) E\left[(s_1 s_2)^{N(B_{12})}\right]$$
$$= e^{\rho |B_1|(s_1-1)} e^{\rho |B_2|(s_2-1)} e^{\rho |B_{12}|(s_1 s_2 - 1)} \,,$$

where $|A|$ = "volume" of A. Now $|A_1| = |B_1| + |B_{12}|$ and $|A_2| = |B_2| + |B_{12}|$ so that

$$G(s_1, s_2) = e^{\rho |A_1|(s_1-1)} e^{\rho |A_2|(s_2-1)}$$
$$\times \exp\left[-\rho |A_1 A_2|(s_1-1) - \rho |A_1 A_2|(s_2-1) + \rho |A_1 A_2|(s_1 s_2 - 1)\right]$$

$$= e^{\rho|A_1|(s_1-1) + \rho|A_2|(s_2-1)}$$
$$\times \exp\left\{\rho|A_1A_2|\left[\underbrace{s_1s_2 - 1 - s_1 + 1 - s_2 + 1}_{\underbrace{(s_1s_2 - s_1)}_{s_1(s_2-1)} - (s_2-1)}\right]\right\}$$

$$= \exp\left[\rho|A_1|(s_1-1) + \rho|A_2|(s_2-1) + \rho|A_1A_2|(s_1-1)(s_2-1)\right]$$

$\mathrm{cov}(N(A_1), N(A_2))$

$= \mathrm{cov}(N(B_1) + N(B_{12}), N(B_2) + N(B_{12}))$

$= \mathrm{cov}(N(B_{12}), N(B_{12})) = \mathrm{Var}(N(B_{12})) = \rho|A_1A_2|$

<u>Note</u> $\mathrm{cov}\left(\sum a_i X_i, \sum b_j Y_j\right) = \sum_{i,j} a_i b_j \,\mathrm{cov}(X_i, Y_j)$

<u>or</u>

$$\mathrm{cov}(N(A_1), N(A_2)) = E(N(A_1)N(A_2)) - \underbrace{E(N(A_1))}_{\rho|A_1|}\underbrace{E(N(A_2))}_{\rho|A_2|}$$

$$\frac{\partial^2}{\partial s_1 \partial s_2} E\left(s_1^{N(A_1)} s_2^{N(A_2)}\right) = E\left[N(A_1)N(A_2)\, s_1^{N(A_1)-1} s_2^{N(A_2)-1}\right]$$

Now set $s_1 = s_2 = 1$ to get $E(N(A_1)N(A_2))$

So in our case

$$\frac{\partial G}{\partial s_2} = \left(\exp\Big[\overbrace{\rho|A_1|(s_1-1)+\rho|A_2|(s_2-1)+\rho|A_1A_2|(s_1-1)(s_2-1)}^{c}\Big]\right) \times \left(\rho|A_2| + \rho|A_1A_2|(s_1-1)\right)$$

&

$$\frac{\partial^2 G}{\partial s_1 \partial s_2} = \left(\exp[c]\right)\left(\rho|A_1A_2|\right) + \left(\exp[c]\right)\left(\rho|A_1| + \rho|A_1A_2|(s_2-1)\right)\left(\rho|A_2| + \rho|A_1A_2| \times (s_1-1)\right)$$

$s_1 = s_2 = 1 \Rightarrow c = 0$ so that

$$\left.\frac{\partial^2 G}{\partial s_1 \partial s_2}\right|_{s_1, s_2 = 1} = \rho|A_1A_2| + \rho|A_1|\rho|A_2|$$

$$\therefore \quad E[N(A_1)N(A_2)] = \rho|A_1A_2| + \rho|A_1|\rho|A_2|$$

and

$$\mathrm{cov}(N(A_1), N(A_2)) = \rho|A_1A_2| + \rho|A_1|\rho|A_2| - \rho|A_1|\rho|A_2|$$

$$= \rho|A_1A_2|$$

p65 #6

$$X = \sum_j N_j$$

$$\Rightarrow G_X(s) = E\left(s^{\sum_j N_j}\right)$$

$$= \prod_j E\left(s^{j N_j}\right)$$

$$= \prod_j E\left[(s^j)^{N_j}\right] = \prod_j e^{\lambda_j (s^j - 1)}$$

$$= \exp\left[\sum_j \lambda_j (s^j - 1)\right]$$

p 74 #1

Let X be memoryless (ageless) & assume $X \in \{1, 2, \dots\}$. We have

$$P(X = r+s \mid X > r) = P(X = s) \quad , s = 1, 2, \dots$$

(this is the lack of memory property in discrete time)

So

$$\frac{P(X > r, X = r+s)}{P(X > r)} = P(X = s)$$

But $\{X > r, X = r+s\} = \{X = r+s\}$ as

$$\{X = r+s\} \subset \{X > r\}$$

Hence $P(X=r+s) = P(X=s)\,P(X>r)$, $\forall r \geq 0,\ s>0$

$\Rightarrow \quad P(X=r+s+k) = P(X=s+k)\,P(X>r)$, $k=1,2,\cdots$

$\Rightarrow \quad \sum_{k=1}^{\infty} P(X=r+s+k) = \sum_{k=1}^{\infty} P(X=s+k)\ P(X>r)$

$\Rightarrow \quad P(X>r+s) = P(X>s)\,P(X>r)$

$\Rightarrow \quad \bar{F}(r+s) = \bar{F}(s)\,\bar{F}(r)$, $\forall r,s \geq 0$ $\quad (*)$

where $\bar{F}(x) = P(X>x) = 1 - \underbrace{P(X \leq x)}_{df}$

$(*) \Rightarrow \quad \bar{F}(k) = (\underbrace{\bar{F}(1)}_{\alpha})^{k}$

$\Rightarrow P(X=k) = P(X>k) - P(X>k-1) = \alpha^{k-1}(1-\alpha)$

which are the geometric probabilities.

p78 #3 Let $X \sim \text{gamma}(r, \lambda)$. Then

$$f(x) = \frac{\lambda^r e^{-\lambda x} x^{r-1}}{\Gamma(r)}, \quad x > 0,$$

where $\Gamma(r) = \int_0^\infty e^{-\lambda x} x^{r-1} dx$, $r > 0$. (Note that $X \overset{d}{=} \frac{Z}{\lambda}$ where $Z \sim \text{gamma}(r) = \text{gamma}(r, 1)$ + $\overset{d}{=}$ means has the same dist'n.) The mgf is

$$E(e^{tX}) = \int_0^\infty \frac{\lambda^r e^{-\lambda x} x^{r-1}}{\Gamma(r)} e^{tx} dx$$

$$= \int_0^\infty \frac{\lambda^r e^{-x(\lambda - t)} x^{r-1}}{\Gamma(r)} dx$$

$$= \frac{\lambda^r}{(\lambda - t)^r} \int_0^\infty \frac{(\lambda - t)^r e^{-x(\lambda - t)} x^{r-1}}{\Gamma(r)} dx$$

$$= \left(\frac{\lambda}{\lambda - t} \right)^r, \quad t < \lambda$$

as the expression in $\int_0^\infty$ is the $\text{gamma}(r, \lambda - t)$ pdf p 78 #4

$$|A(r)| = K_d r^d$$

so that

$$P(S_1 > s) = P(\text{no points in } A(s))$$
$$= e^{-\rho A(s)}$$

$$\therefore \quad f(s) = -\frac{d}{ds} P(S_1 > s)$$
$$= e^{-\rho K_d s^d} (\rho K_d d) s^{d-1}, \quad s > 0$$

Now

$$|A(S_1)| = K_d S_1^d$$

$$\Rightarrow E(|A(S_1)|) = \int_0^\infty K_d s^d (\rho K_d d) s^{d-1} e^{-\rho K_d s^d} ds$$

$$\stackrel{u = K_d s^d}{=} \int_0^\infty u \rho e^{-\rho u} du = 1/\rho$$

In fact $|A(S_1)|$ is exponential(ρ) as can be seen by evaluating, for $s_1 > 0$,

$$P(|A(S_1)| > s_1) = P(K_d S_1^d > s_1) = P\left(S_1 > \left(\frac{s_1}{K_d}\right)^{\frac{1}{d}}\right)$$

$$= \int_{\left(\frac{s_1}{K_d}\right)^{1/d}}^{\infty} \rho K_d d s^{d-1} e^{-\rho K_d s^d} ds$$

$$\underset{u = K_d s^d}{=} \int_{s_1}^{\infty} \rho e^{-\rho u} du = e^{-\rho s_1}$$

so that $|A(S_1)| \sim \underline{\text{exponential}}(\rho)$

Assignment#4 Soluti

1. $E(Y|A) = E(Y I_A)/P(A)$.

- $E(1|A) = E(I_A)/P(A) = 1$
- $Y > 0 \Rightarrow Y \geq 0 \Rightarrow E(Y I_A) \geq 0 \Rightarrow E(Y|A) \geq 0$
- $E(cY|A) = E(cY I_A)/E(I_A) = c\,E(Y I_A)/E(I_A) = c\,E(Y|A)$
- $E(X+Y|A) = \dfrac{E((X+Y) I_A)}{E(I_A)} = \dfrac{E(X I_A)}{E(I_A)} + \dfrac{E(Y I_A)}{E(I_A)}$

$$= E(X|A) + E(Y|A)$$

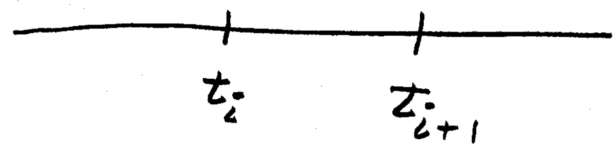
- $Y_n \uparrow Y \Rightarrow Y_n I_A \uparrow Y I_A \Rightarrow \dfrac{E(Y_n I_A)}{E(I_A)} \rightarrow \dfrac{E(Y I_A)}{E(I_A)}$

so that $E(Y|A) = \lim E(Y_n|A)$

#5 $X = \sum_{i=1}^{n} I_i$, where $I_i \sim \text{Bernoulli}\left(\frac{1}{n}\right)$

$$\Rightarrow E(X) = \sum_{i=1}^{n} E(I_i) = \sum_{i=1}^{n} \frac{1}{n} = 1$$

#7



$$N(t_{i+1}) = N(t_i) + \left[N(t_{i+1}) - N(t_i)\right]$$

ind

$\Rightarrow$ conditional on $N(t_i) = m$

$$N(t_{i+1}) \overset{d}{=} m + \text{Poisson}\left(\lambda(t_{i+1} - t_i)\right) \text{ rv}$$

$$\Rightarrow \quad E\left(N(t_{i+1}) \mid N(t_i) = m\right) = m + \lambda(t_{i+1} - t_i)$$

$$\Rightarrow \quad E\left(N(t_{i+1}) \mid N(t_i)\right) = N(t_i) + \lambda(t_{i+1} - t_i)$$

p54#3

Let $X_1, \cdots, X_n$ be iid uniform$(\{1, \cdots, M\})$. Let $U = X_{(n)}$, the largest order statistic. The probability

$$P(X_{(n)} \le x) = P(X_1 \le x, \cdots, X_n \le x)$$
$$= P(X_1 \le x) \cdots P(X_n \le x)$$
$$= \left[P(X_1 \le x)\right]^n$$

is easy to calculate. [Aside $P(X_{(1)} > x) = \left[P(X_1 > x)\right]^n$ is also simple.] In this problem we want to to calculate $P(X_{(m)}=x)$, $x=1,\cdots,M$. This is just

$$P(X_{(m)}=x) = P(X_{(m)}\le x) - P(X_{(m)}\le x-1)$$
$$= [P(X_1\le x)]^m - [P(X_1\le x-1)]^m$$
$$= \left(\frac{x}{M}\right)^m - \left(\frac{x-1}{M}\right)^m \quad , x=1,\cdots,M$$

<u>#4</u> We will verify

$$E[Yh(X)] = E[E(Y|X)\,h(X)]$$

in the cts case. The discrete case is identical.

$$E[E(Y|X)\,h(X)] = \int_{-\infty}^{\infty} E(Y|X=x)\,h(x)\,f(x)\,dx$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\,f(y|x)\,dy\;h(x)\,f(x)\,dx$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\,f(y|x)\,f(x)\,h(x)\,dy\,dx$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\,h(x)\,f(x,y)\,dy\,dx$$
$$= E[Y h(X)]$$

Distribution of "the Range" in a Poisson Process

Let $\{N(t) \mid t \geq 0\}$ be a Poisson process of rate λ on $t \geq 0$. It is known that $N(T) = N$ and we wish to find the pdf of $W = T_N - T_1$, where $T_1 < T_2 < \cdots < T_N$ are the N points in $[0, T]$.

Sol'n #1 (appeal to order statistics)

It is known that $T_1, \cdots, T_N$ have the same dist'n as that of the order statistics from a sample of size N from a uniform $([0,T])$. It thus follows that $\frac{T_1}{T}, \cdots, \frac{T_N}{T}$ are distributed as the order statistics for a sample of N from the uniform $([0,1])$. Set

$$V = \frac{T_N}{T} - \frac{T_1}{T} = Y - X,$$

where $Y = T_N/T$ and $X = T_1/T$. Now derive the joint pdf of X & Y as before. This yields

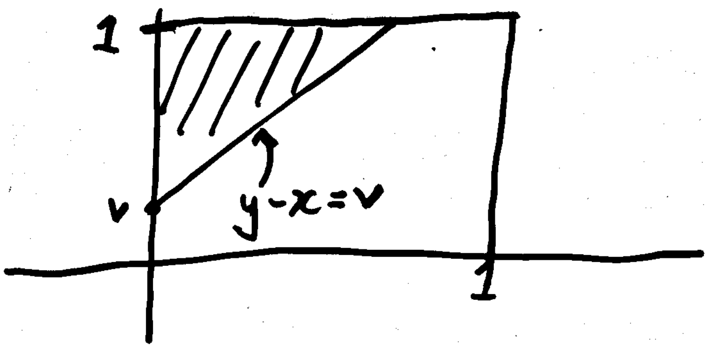
$$f(x,y) = N(N-1)(y-x)^{N-2}, \quad 0 \leq x < y \leq 1$$

Let $v \in [0,1)$ and consider $\bar{F}_V(v) = P(V > v)$

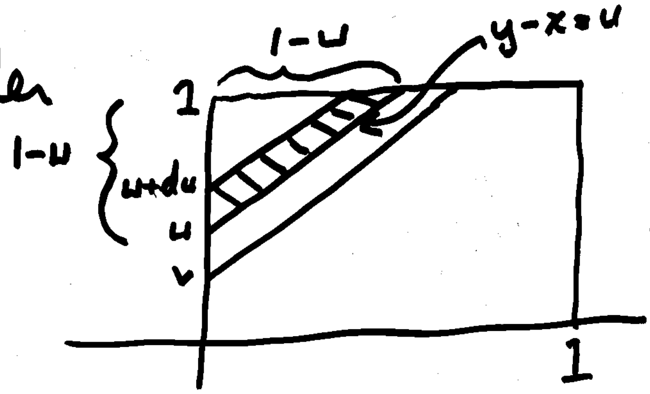
We have

$$\bar{F}_V(v) = \iint_{///} f(x,y)\,dx\,dy \quad ,$$

where /// is as in the picture



Consider



The area of \\\\ $\approx$ $(1-u)\,du$ and in \\\

$$f(x,y) \approx N(N-1)\,u^{N-2}$$

Hence

$$\bar{F}_V(v) = \int_v^1 N(N-1)u^{N-2}(1-u)\,du$$

so that

$$f_V(v) = -\bar{F}_V{}'(v) = \frac{d}{dv}\int_1^v N(N-1)u^{N-2}(1-u)\,du$$

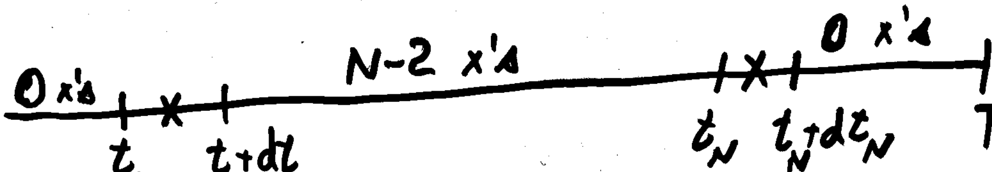
$$f_V(v) = N(N-1)\, v^{N-2}(1-v) \quad , \; 0 < v < 1$$
$$= 0 \quad , \; ow$$

Now $W = TV$ so that

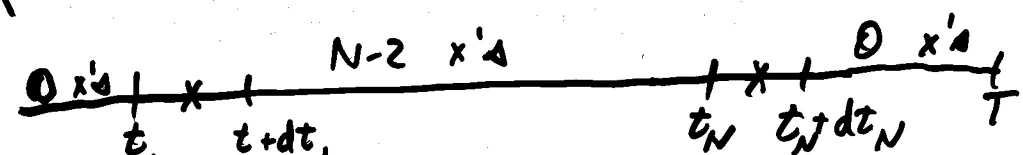
$$f_W(w) = \frac{N(N-1)}{T}\left(\frac{w}{T}\right)^{N-2}\left(1-\frac{w}{T}\right) \quad , \; 0 < w < T$$
$$= 0 \quad , \; ow$$

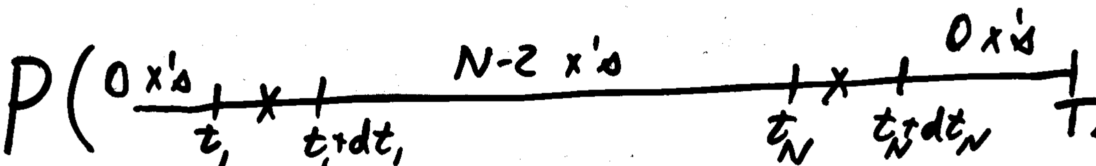
<u>Sol'n #2</u> The joint pdf of T_1 & T_N, conditional on $N(T) = N$, satisfies

$$f(t_1, t_N)\,dt_1\,dt_N \approx P\big(T_1 \in (t_1, t_1 + dt_1),\, T_N \in (t_N, t_N + dt_N)\big)$$

$\approx P\Big($


$\Big| N(T) = N \Big)$

$= \dfrac{P\Big(\text{[diagram]}, N(T)=N\Big)}{P(N(T) = N)}$


$= \dfrac{P\Big(\text{[diagram]}\Big)}{P(N(T) = N)}$


Now use the Poisson process properties to reduce this to

$$\frac{\left(e^{-\lambda t_1}\right)\left(e^{-\lambda dt_1}\lambda dt_1\right)\left(e^{-\lambda(t_N-t_1-dt_1)}\dfrac{[\lambda(t_N-t_1-dt_1)]^{N-2}}{(N-2)!}\right)\left(e^{-\lambda dt_N}\lambda dt_N\right)\left(e^{-\lambda(T-t_N-dt_N)}\right)}{e^{-\lambda T}\cdot(\lambda T)^N/N!}$$

$$= N(N-1)\left(\frac{dt_1}{T}\right)\left(\frac{t_N-t_1-dt_1}{T}\right)^{N-2}\left(\frac{dt_N}{T}\right)$$

and so

$$f(t_1,t_N) = \frac{N(N-1)}{T^2}\left(\frac{t_N-t_1}{T}\right)^{N-2}, \quad 0 \le t_1 < t_N \le T$$
$$= 0, \quad ow$$

Now set $X = \frac{T_1}{T}$, $Y = \frac{T_N}{T}$ to get the joint pdf of X & Y as

$$f(x,y) = N(N-1)(y-x)^{N-2}, \quad 0 \le x < y \le 1$$
$$= 0, \quad ow$$

The derivation of $V = Y - X$'s pdf & that of $W = TV$ then is as in Sol'n #1