Lecture 5

Apples into barrels

\[ \square \square \square \ldots \square \]

M barrels
N apples
Apples "identical"

2 cases < all different

\[ \rightarrow \quad M^N \quad \leftarrow \]

Fewer
N “points” into M cells
If done uniformly and independently then
\[ P(Y_1 = k_1, \ldots, Y_M = k_M) = \binom{N}{k_1, \ldots, k_M} \cdot \frac{1}{M^N} \]
which is the multinomial \((N; \frac{1}{M}, \ldots, \frac{1}{M})\) or Maxwell-Boltzmann distribution (occupancy statistics).

Remark
(i) If the points are indistinguishable then the number of arrangements of the points in the cells is
\[ \sum_{k_1, \ldots, k_M} \binom{N}{k_1, \ldots, k_M} = (1 + \ldots + 1)^N = M^N \]
Of course different arrangements may lead to the same counts.
(ii) \(Y_i \sim \text{binomial}(N, \frac{1}{M}) \Rightarrow \text{Poisson}(\mu)\)
\[ \text{if } \frac{N}{M} \approx \mu + N \text{ is large.} \]
If the points are indistinguishable then the \# of arrangements will be fewer. We can count them as follows:

\[
\begin{array}{cccc}
\text{x} & \text{x} & \text{x} & \text{x} \\
\end{array}
\]

\[N-x's\]

\[M-1 = \text{froze longer than the end lines}\]

As we move the 1's + x's around we obtain the different ways of putting the N points into the M cells (we are only keeping track of the counts in each cell). This is identical to moving M-1 1's + N 0's around in N+M-1 cells and asking for the \# of arrangements! So we obtain

\[(N+M-1)_{N} \times (N+M-1)_{M-1}\]

Notice that each arrangement corresponds to counts in each of the M cells! If equal probability is assigned to each then we have the "Bose-Einstein statistics"
In particular
\[ P(Y_i = k_i) = \binom{N-k_i+M-2}{M-2} / \binom{N+M-1}{M-1} \]
Let $x > 0$ then

$$F_n(x) = P(S_n > x) = P(N(x) \leq n - 1)$$

$$= \sum_{k=0}^{n-1} e^{-\lambda x} \frac{(\lambda x)^k}{k!}$$

Now calculate $F_n'(x)$ to get the pdf of a gamma (check this).

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**Extra material (basic conditioning, a review)**

**Def'n** If $P(A) > 0$ define $E(Y|A) = \frac{E[Y I(A)]}{E[I(A)]]}.$

It is easily seen that, for fixed $A$, $E(\cdot | I(A))$ satisfies the Axioms for $E$ (verify this).

**Def'n** $P(B|A) = E(I(B) | I(A))$

Of course $P(B|A) = P(BA) / P(A)$ as before.
If \( X \) is discrete then we set
\[
\eta(x) = E(Y|X = x)
\]
and
\[
E(Y|X) = \eta(X)
\]
If \( Y \) is also discrete then it is easily verified
\[
E(Y) = E[E(Y|X)]
\]
and
\[
E(Y|X_1) = E[E(Y|X_1, X_2)|X_1]
\]
A further property is
\[
E[Y g(X)] = E[E(Y|X) g(X)], \quad \forall \text{ real } g
\]
These last three properties are true in general and are easily verified when \((Y, X)\)
is its with PDF \( f(y, x) \).

Finally, it can be shown that \( E(Y|X) \) satisfies (see Th 5.3.1)
\[
E(Y - E(Y|X))^2 = \min_{h \in \mathcal{H}} E(Y - h(X))^2
\]
Application of the uniform + Bernoulli / Indicator

$N$ - towns, at most 1 road between them, roads of any 2 towns

In fact, there is always a collection of towns, at least half of roads lead into it.

Solve: Go to each town and toss a fair coin. If $H$ put town into a collection $S$. $S$ is a random collection of towns. Note that there are $2^N$ possible "values" that $S$ can be. Let $X = \#$ of roads leading into $S$ from outside. We need to show there is

\[ \frac{2}{3} \text{ of roads go into the dotted collection} \]
a possible value of $X \geq m/2$. Since $X$ is a (positive) counting rv this will hold if $E(X) \geq m/2$. Now, let

$$I_j = I(\{\text{road } j \text{ leads into } S\})$$

Then

$$X = \sum_{j=1}^{m} I_j$$

and

$$E(X) = \sum_{j=1}^{m} E(I_j) = m \cdot P(\text{road } 1 \text{ leads into } S)$$

$$= m \cdot P(\text{one } H \text{ and one } T \text{ for the } 2 \text{ tosses})$$

$$= m/2$$

Let $\{N(t) : t \geq 0\}$ be a Poisson process of rate $\lambda$ on $t \geq 0$. Suppose you know $N(t_0) = m$. Call the times of the points $T_1 < T_2 < \cdots < T_m$
We want the pdf \( f_X \).

**Approach**
Use def \( F \) or \( F = 1 - F \)

**Aside**

\[
F(x) = P(X \leq x), \quad \forall x \in \mathbb{R}
\]


For a cts rv \( X \)

\[
F(x) = \int_{-\infty}^{x} f(t) \, dt
\]

\[=\] at continuity pts

\[
F'(x) = f(x)
\]

In fact we could take \( F' \) to be an equivalent pdf.

\[
P(b) - P(a) = P(a < X \leq b)
\]

\[
\begin{align*}
F(a) & \quad \Rightarrow \quad F(a) = P(X \leq a) + P(a < X \leq b) = P(X \leq b) - P(a < X \leq b) = \int_{a}^{b} f'(t) \, dt \quad \boxed{FLAC} \\
& \quad \Rightarrow \quad F(b) - F(a) = P(a < X \leq b)
\end{align*}
\]
Remark. If \( F \) is cts it is possible for \( F'(x) = 0 \) for almost every \( x \) (places where not true have length 0), so in that case \( \int_{-\infty}^{\infty} F'(x) \, dx = 0 \).

\( F'(x) = -f(x) \). So knowing \( F \Rightarrow \) know the dist \( \mu \).

By the way,

\[ x_1 < x_2 \Rightarrow P(X \leq x_1) \leq P(X \leq x_2) \]

\[ \therefore \{X \leq x_1\} \Rightarrow \{X \leq x_2\} \]

\[ \therefore F \text{ is increasing.} \]

\[ \therefore x_n \downarrow a \Rightarrow \{X \leq x_n\} \downarrow \{X \leq a\} \]

\[ \Rightarrow P(X \leq x_n) \to P(X \leq a) \]

(Continuity property of \( P \)).

\( \therefore F \text{ is right cts} \)
$$x_n \uparrow \infty \Rightarrow \{X \leq x_n\} \uparrow \{X < \infty\}$$
$$\Rightarrow F(x_m) \to P(X < \infty) = 1$$

$$x \to \infty \Rightarrow F(x) \to 1$$
$$F(\infty) = 1$$

$$x_n \downarrow -\infty \Rightarrow \exists X \leq x_n \exists \downarrow \emptyset$$
$$\Rightarrow F(x_m) \to 0$$

$$x \to -\infty \Rightarrow F(x) \to 0$$
$$F(-\infty) = 0$$

Note: $x_n \uparrow a \Rightarrow \{X \leq x_n\} \uparrow \{X < a\}$

$$\Rightarrow \lim_{x \uparrow a} F(x) = P(X < a) \neq F(a)$$

$$P(X = a) = F(a) - \lim_{x \uparrow a} F(x)$$

(2) Knowing $F \Rightarrow$ know $P(a \leq X \leq b)$

(2) Knowing $F \Rightarrow$ know the distribution
Back to our Poisson process.

\[ F_m(x) = P(T_m \leq x) = P(T_m \leq x | N(x, t_0) = m) \]

\[ = P(N((x, t_0]) = 0 | N(t_0) = m) \]

\[ = P(N((x, t_0]) = 0, N(t_0) = m) \]

\[ \frac{P(N(t_0) = m)}{P(N(t_0) = m)} \]

\[ = \frac{P(N(x) = m, N((x, t_0]) = 0)}{P(N(t_0) = m)} \]

\[ = \frac{P(N(x) = m) P(N((x, t_0]) = 0)}{P(N(t_0) = m)} \]
\[ = P(N(x) = m) \frac{P(N(t_0) = 0)}{P(N(t_0) = m)} \]
\[ = e^{-\lambda x} \frac{(dx)^m}{m!} e^{-\lambda(t_0-x)} \]
\[ = \frac{e^{-\lambda t_0} \frac{(\lambda t_0)^m}{m!}}{m!} \]
\[ = \left( \frac{2x}{t_0} \right)^m \]

\[ \text{ conditional on } N(t_0) = m, \]
\[ F_{(m)}(x) = \left( \frac{2x}{t_0} \right)^m, \quad 0 \leq x \leq t_0 \]
\[ = 1, \quad x > t_0 \]
\[ = 0, \quad \text{otherwise} \]
\[ f(m) = \frac{n}{t_0} \left( \frac{x}{t_0} \right)^{m-1}, \quad 0 \leq x \leq t_0 \]

\[ = 0, \quad \text{otherwise} \]

If \( t_0 = 1 \) then
\[ f(x) = n \cdot x^{m-1}, \quad 0 \leq x \leq 1 \]
\[ = 0, \quad \text{otherwise} \]

\[ \text{Remark: } \quad \mathbb{E}(Y \mid X = x) = \sum_y y \frac{f(y \mid x)}{f(x)} \]
\[ = \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(X = x)} \]
\[ \text{discrete} \]

\[ \mathbb{E}(Y \mid X) = \mathbb{E}(X) \]

\[ \mathbb{E}[\mathbb{E}(Y \mid X)] = \mathbb{E}(X) \]

\[ \mathbb{E}[\mathbb{E}(Y \mid X)] = \mathbb{E}(Y) \]

\[ \mathbb{E}(X \mid X = x) = \mathbb{E}(X) = \int_{-\infty}^{\infty} y f(y \mid x) \, dy \]

\[ \text{cond pdf} \]
stochastic process = collection of random elements

Gaussian process = collection of rv's whose dist'ns are normal

Markov property
{X_t : t \in T}

For each t_0 \in T,

\[ g(\{X_t : t > t_0\} | \{X_t : t \leq t_0\} ) \]

stationary process:
(\(X_{t_1}, \ldots\)) \distr (\(X_{t_1+\Delta}, \ldots\)), \(\Delta\) a

toeplitz matrix

\[ \begin{pmatrix} 0 & 0 & \cdots \\ \vdots & \ddots & \vdots \\ 0 & 0 & 1 \end{pmatrix} \]

elements on 0 same square

\(\cdots\)}
Poisson processes & order stats

**Sample** $X_1, \ldots, X_m$

**Order stats** $X_{(1)} < \cdots < X_{(m)}$

(assume cts)

Poisson process

- throw points $\rightarrow$ Poisson process
- have process $\rightarrow$ some other way to condition $\rightarrow$ throw pts!

So

Let $\{ N(t) : t \geq 0 \}$ be a Poisson process of rate $\lambda$ on $t \geq 0$.

Suppose you know $N(t_0) = m$. Call the times of the points $T_1 < T_2 < \cdots < T_m$

\[
\begin{array}{cccccccc}
0 & T_1 & T_2 & \cdots & T_m & t_0 \\
\end{array}
\]
We want the pdf of $T_n$.

**Approach**

Use the definition $F$ or $ar{F} = 1 - F$

**Aside**

$F(x) = P(X \leq x), \forall x \in \mathbb{R}$

For a cts rv $X$

$$F(x) = \int_{-\infty}^{x} f(t) \, dt$$

$\Rightarrow$ at continuity pts

$$F'(x) = f(x)$$

In fact we could take $F'$ to be an equivalent pdf.

$$F(b) - F(a) = P(a < X \leq b)$$

$$\leq P(X \leq a) + P(a < X \leq b) = P(X \leq b)$$

$$= \int_{a}^{b} F'(t) \, dt$$

**FLAC**
Back to our Poisson process.

$$F_{(m)}(x) = P(T_m \leq x) \{ = P(T_m \leq x \mid N(X) = m)}$$

\[ m \times \frac{T_m}{X} \leq x \leq T_0 \]

\[ F_{(m)}(x) = \left( \frac{2c}{t_0} \right)^m \quad , \quad 0 \leq x \leq t_0 \]

= 1 \quad , \quad x > t_0 \]

= 0 \quad , \quad o.w.

Q. so
\[ f(m)(x) = \frac{m}{\tau_0} \left( \frac{x}{\tau_0} \right)^{m-1}, \quad 0 \leq x \leq \tau_0 \]

\[ = 0, \quad \text{otherwise} \]

If \( \tau_0 = 1 \) then
\[ f(m)(x) = m \cdot x^{m-1}, \quad 0 \leq x \leq 1 \]

\[ = 0, \quad \text{otherwise} \]

As we will see this is what we get when looking at the order statistics from a uniform.

First interarrival times ind?

Conditional pdf/ pf/mgf/ pgf/df = unconditional

\[ \implies \text{ind} \]
eg. Poisson process, rate $\lambda$ on $t > 0$

We know $X_i \sim \text{exponential}(\lambda)$

Look at

$$P(X_2 > x_2 \mid X_1 = x_1)$$

$$= P\left(N\left((x_1, x_1 + x_2)\right) = 0 \mid X_1 = x_1\right)$$

= $\text{e}^{-\lambda x_2}$

which is the tail probability of an
\[ \text{exponential}(1). \]

\[ X, X_2 \text{ are independent } \Rightarrow \text{ unconditional tail probability if } n \text{ is as above. That is, } X, X_2 \text{ are iid exponential}(1). \text{ Continue to get} \]

\[ X, X_2, \ldots \text{ iid exponential} \]

\[ \text{Order Statistics} \]

\[ X, X_2, \ldots, X_n \text{ iid pdf } f \]

\[ X^{(1)} < \ldots < X^{(n)} \text{ order stats} \]

pdf of \( X^{(2)} \)

\[ \int \frac{(x) \, dx}{(m - 2)} \]

\[ \sim \mathbb{P}\left( \frac{1}{n-1} \sum_{i=1}^{n-1} X_i + \frac{1}{n+1} (m-2) \right) \]
\[
F(x) \sim f(x) dx \sim \overline{F}(x)
\]

\[
\Gamma \left( \frac{m}{n-1}, 1, m-n \right) F(x)^{n-1} f(x) dx \overline{F}(x)^m
\]

\[
f(n)(x) = \binom{m}{n-1} \Gamma \left( \frac{m}{n-1}, 1, m-n \right) F(x)^{n-1} f(x) \overline{F}(x)^m
\]

For a uniform \((0, 1)\)

\[
f(m)(x) = mx^{m-1}, \quad 0 < x < 1
\]