(S, \{\text{events}\}, P) - probability space
A \subset B, (A \implies B), I_A \leq I_B

Note: A \subset B \iff I_A \leq I_B

Proof: Assume A \subset B. We must show
I_A(s) \leq I_B(s), \forall s \in S

So let s \in A. Then I_A(s) = 1. But A \subset B \implies s \in B \implies I_B(s) = 1. Consequently,
I_A(s) \leq I_B(s), \forall s \in A

Now take s \in A^c. Then I_A(s) = 0. Since I_B(s) \geq 0 no matter what s is we get
I_A(s) \leq I_B(s), \forall s \in A^c

\therefore I_A \leq I_B

Now assume I_A \leq I_B. We must show A \subset B.
To see this let \( x \in A \). We must show \( x \in B \).
If \( x \) is in \( \mathcal{I}_B \) then \( I_B(x) = 0 \). But \( I_A(x) = 1 \) \( \Rightarrow I_A(x) \leq I_B(x) \). This can't be so \( x \) must be in \( B \). \( \therefore A \subset B \) \( \subset \) hence \( A \subset B \iff I_A \leq I_B \).

**Calculation using symmetry**

eg Select 2 cards. \( P(\text{both are spades}) \)?

**Sol'n # of "2 card hands"**

\[
\binom{52}{2} = \frac{52!}{2! \cdot 50!} = \frac{52 \times 51}{2 \times 1} \cdot \frac{50!}{2 \times 1 \times 50!}
\]

# of "2 card hands" made up only of spades is \( \binom{13}{2} \)

\( \therefore P(\text{both are spades}) = \frac{\binom{13}{2}}{\binom{52}{2}} \)
Roll a fair die and let $X =$ # of dots.

$P(X \text{ is even}) = \frac{3}{6}$

$P(X \geq 4) = \frac{3}{6}$

Let $B = \{X \text{ is even}\} \land A = \{X \geq 4\}$

Suppose you are told $A$ has occurred; then one would update the probability of $B$ to $\frac{2}{3}$. This is the conditional probability of $B$ given $A$. The usual notation is

$P(B \mid A) = \frac{2}{3}$ here

$\{P_A(B)\}$

**Definition** $P(B \mid A) = \frac{P(AB)}{P(A)}$

Note: For fixed $A$, $P(\cdot \mid A)$ satisfies the Laws of $P$. 
\[ P(AB) = P(A) P(B|A) \]

3. \( A + B \) independent (\( P(AB) = P(A) P(B) \)) gives us \( P(B|A) = P(B) \).

Ex: Back to \( P(\text{both are spades}) \) ?

Let:

\[ \begin{align*}
A &= \{ \text{spade on 1st draw} \} \\
B &= \{ \text{spade on 2nd draw} \}
\end{align*} \]

\[ P(AB) = P(\text{both are spades}) \]

\[ = P(A) \ P(B|A) \]

\[ = \frac{13}{52} \times \frac{12}{51} \]

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Bayes formula type problems

```
H - P
H - A
T - q
```

\[ \begin{array}{c|cc}
m_1 & b's & w'ds \\
m_2 & b's & w'ds \\
\hline
\text{Hat #1} & \text{select a chip} & \\
\text{Hat #2} & \text{select a chip} & \\
\end{array} \]
Let $N_1 = \# \text{ of chips in Hat 1}$

+ $N_2 = \# \text{ of chips in Hat 2}$

$P(b) = P(b \text{ and } H) + P(b \text{ and } T)$

$= P(H) P(b \mid H) + P(T) P(b \mid T)$

$= p \times \frac{m_1}{N_1} + q \times \frac{m_2}{N_2}$

$P(H \mid b) = \frac{P(H \text{ and } b)}{P(b)} = \frac{P(b \text{ and } H)}{P(b)}$

$X : S \rightarrow \mathbb{R}$

Expectation, expected value, mean of $X$

$E(X)$
We must have
\[ E(A) = P(A) \] (*)

We would like \( E \) to have the following properties:

- \[ E(1) = 1 \] (I)
- \( X \geq 0 \Rightarrow E(X) \geq 0 \) (II)
- \[ E(cX + dY) = cE(X) + dE(Y) \] (III)
- \[ E(\sum_{k=1}^{\infty} X_k) = \sum_{k=1}^{\infty} E(X_k) \geq 0 \] (IV)

**Theorem:** There exists a unique \( E : \{\text{r.v.'s}\} \rightarrow \mathbb{R} \) satisfying (*)

\[ + (\text{II}) \rightarrow (\text{IV}). \]
We know
\[ E(X_1 + X_2) = E(X_1) + E(X_2) \]

**Proposition** \[ E(X_1 + X_2 + \cdots + X_N) = E(X_1) + E(X_2) + \cdots + E(X_N), \]
\[ N \geq 2 \]

**Proof** We know the result is true for \( N = 2 \).
Assume it's true for \( N = n \geq 2 \). Now take \( N = n + 1 \). Look at
\[ E(X_1 + X_2 + \cdots + X_n + X_{n+1}) \]
\[ = E \left[ (X_1 + X_2 + \cdots + X_n) + X_{n+1} \right] \]
\[ = E(X_1 + X_2 + \cdots + X_n) + E(X_{n+1}) \]
\[ = E(X_1) + \cdots + E(X_n) + E(X_{n+1}) \]

so the result holds for \( N = n + 1 \) and hence \( N \geq 2 \) by induction. 
\[ \text{QED} \]
discrete rv's

A rv $X$ is **discrete** if its range is countable. Suppose the range is the set $\{x_1, x_2, \ldots\}$. Now let $A_k = \{X = x_k\}$, $A_1, A_2, \ldots$ partition $S$.

$$X = x_1 I_{A_1} + x_2 I_{A_2} + \ldots$$

$$g(X) = g(x_1) I_{A_1} + g(x_2) I_{A_2} + \ldots$$

"Hence"

$$\mathbb{E}[g(X)] = g(x_1) \mathbb{E}(I_{A_1}) + g(x_2) \mathbb{E}(I_{A_2}) + \ldots$$

$$= g(x_1) P(X = x_1) + g(x_2) P(X = x_2) + \ldots$$

$$= \sum_{x \in \mathbb{X}} g(x) P(X = x)$$
Call \( f(x) = \Pr(X = x) \)

the probability function (pf).

Note

\[
\begin{align*}
&\quad f(x) \geq 0 \\
&\quad \sum_{x} f(x) = 1
\end{align*}
\]

conditions for a function to be a pf.

\( E(X) \) is also called the mean — \( \mu \)

\( E(X^2) \) is called the 2nd moment

\( E[(X-\mu)^2] \) is the variance of \( X \rightarrow \text{Var}(X) \)

\( \text{SD}(X) = \sqrt{\text{Var}(X)} \rightarrow \sigma \)

Note \( \text{Var}(X) = E[(X-\mu)^2] \)

\[
= E(X^2 + \mu^2 - 2\mu X) \\
= E(X^2) + \mu^2 - 2\mu E(X) = E(X^2) - \mu^2
\]