Bayesian Inference for Conditional Copula models with Continuous and Binary Responses

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Outline

Copulas

- Copula functions are used to model dependence between continuous random variables.
- (Sklar,'59) If Y_1, Y_2 are continuous r.v.'s with distribution functions (df) F_1, F_2, there exists an unique copula function $C : [0,1] \times [0,1] \to [0,1]$ such that

$$F_{12}(t,s) = \Pr(Y_1 \leq t, Y_2 \leq s) = C(F_1(t), F_2(s)).$$

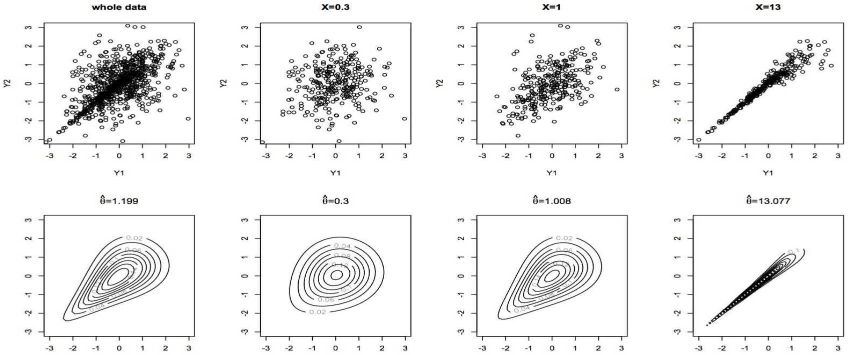
- C is a distribution function on $[0,1]^2$ with uniform margins.
- The copula bridges the marginal distributions with the joint distribution.

Inference for Copula Models

- Early on: abundance of theoretical developments: construction of new copula families and connections with dependence concepts (NA, PQD/NQD, PRD/NRD, etc). Joe ('97), Nelsen ('06).
- Statistical inference for classical parametric copulas
  - Joint Maximum Likelihood: numerical methods
  - Two-stage approach: Joe (JMVA, '05)
  - Semiparametric approach: Genest, Khoudi and Rivest (Bmka, '95)
- Copula selection and goodness-of-fit (Berg, Eur. J. of Finance, '09).
- Emphasis is placed on copula applications in statistics.
- Today: Dynamic regimes of dependence.

Example: Blood pressure

Example It is known that there is a dependence between blood pressure (BP) and body mass index (BMI). What if the dependence varies with subject's age? Can we still use copulas?



Example: Twin Weight

- Twin live births in which both babies survived in the first year of life with mothers of age between 18 and 40.
- Study the dependence between the birth weights of twins, BW_1 and BW_2.
- The gestational age, GA, is an important factor for prenatal growth.
- How does GA influence the dependence between BW_1 and BW_2?

Example: Smoking cessation

- The smoking cessation study of Liu, Daniels and Marcus (JASA '09) :

  Q = smoking cessation (0=No, 1=Yes)
  W = weight change
  X = time spent exercising

- Does exercise weaken the association between smoking status and weight gain?

Conditional Copulas

- The conditional copula of $(Y_1, Y_2)|X = x$, is the conditional joint distribution function of $U = F_{1|X}(Y_1|x)$ and $V = F_{2|X}(Y_2|x)$ given $X = x$ (Patton, Int'l Econ. Rev. '06).
- Consider a random sample $\{x_i, y_{1i}, y_{2i}\}_{1 \le i \le n}$ and suppose $F_{1|X}$ and $F_{2|X}$ are the unknown marginal conditional cdf's.
- The parametric conditional copula model assumes

$$(Y_{1i}, Y_{2i})|X = x_i \sim C(F(Y_{1i}|x_i), F(Y_{2i}|x_i)|\theta(x_i)).$$

- Marginals and copula are conditional on the same variables.

General comments

- Conditional copulas (CC) broaden the range of applications.
  (i) more realistic use of copula models in regression settings.
  (ii) improve the interpretability and understanding of covariate-varying dependence structures.
- Flexible models for the relationship between the association measure (copula parameter, Kendall's tau, etc) and covariate(s) are needed.

A second motivation

- Joint models for high dimensional data.
- The joint distribution of (U_1, U_2, U_3) is modelled using the pair copula model is

$$c(u_1, u_2, u_3) = c_{12}(u_1, u_2)c_{23}(u_2, u_3)c_{13|2}(u_{1|2}, u_{3|2}; u_2)$$

  where $u_{k|2} = Pr(U_k \leq u_k | U_2 = u_2)$.
- Acar, Genest and Neslehova (JMVA, '12) show that wrongly assuming $c_{13|2}$ is the same for all u_2 leads to biased estimators.

Dependence models for CC's in Linear Regression Models

- V_1, V_2 are continuous r.v.'s, $V_i \sim \mathcal{N}(X\beta_i, \sigma_i^2)$, for $i = 1, 2$.
- Jointly,

$$\begin{aligned} f(V_1, V_2|X) &= \prod_{i=1}^{2} \frac{1}{\sigma_i} \phi\left(\frac{V_i - X\beta_i}{\sigma_i}\right) \\ &\times \quad c^{(1,1)} \left\{ \Phi\left(\frac{V_1 - X\beta_1}{\sigma_1}\right), \Phi\left(\frac{V_2 - X\beta_2}{\sigma_2}\right) \middle| \theta(X) \right\}, \end{aligned}$$

  where $c^{(a,b)}(u, v|\theta) = \partial^{a+b} C(u, v|\theta)/\partial u^a \partial v^b$, for all $0 \leq a, b \leq 1$.
- Choose g such that $g(\theta(x_i)) = \eta(x_i)$, where $\eta : \mathbf{R} \to \mathbf{R}$ is the calibration function in inferential focus.

Calibration Model

- A modified version of the cubic spline model of Smith and Kohn (J. Econometrics, '96), Fan et al. (JCGS, '10)

$$\eta(z) = \sum_{j=0}^{3} \alpha_j z^j + \sum_{k=1}^{K} \psi_k (z - \gamma_k)_+^3.$$

- Number and location of knots is influential.

Why a Bayesian approach?

- Joint modelling avoids the propagation of errors.
- Samples from $\pi(\omega|\mathcal{D})$ lead to finite sample variance estimates, pointwise credible regions, computation of model selection criteria.
- Allows data-driven choice of knots location.
- Can use Bayesian model averaging to account for model uncertainty.
- Requires careful examination of the prior's influence.

Bayesian Curve Fitting with Cubic Splines

- Alternative to reversible-jump MCMC approach.
- Partition the range of X into K_{max} intervals, I_k, and introduce auxiliary variables

$$\zeta_k = \begin{cases} 1 & \text{if there is a knot } \gamma_k \text{ in } I_k \text{ and } \psi_k \neq 0 \\ 0 & \text{if there is no knot in } I_k \text{ and } \psi_k = 0 \end{cases}$$

- $\eta(z) = \sum_{j=0}^{3} \alpha_j z^j + \sum_{k=1}^{K_{max}} \psi_k \zeta_k (z - \gamma_k)^3{}_+,$

Bayesian Curve Fitting with Cubic Splines (cont'd)

- Let $|\zeta| = \sum_{k=1}^{K_{max}} \zeta_k$ be the number of knots that are used in the model and set:

  (i) $\lambda \sim \text{Binomial}(K_{max}, p = 0.5)$.

  (ii) $p(|\zeta| \mid \lambda) \propto \frac{\lambda^{|\zeta|}}{|\zeta|!} \mathbf{1}_{\{|\zeta| \leq K_{max}\}}$

  (iii) $p(\zeta \mid |\zeta|) = \binom{K_{max}}{|\zeta|}^{-1}$, all configurations are equally likely.

- $p(\zeta) = p(\zeta \mid |\zeta|)p(|\zeta|)$.

Bayesian Curve Fitting with Cubic Splines (cont'd)

- Prior distributions:

$$
\begin{aligned}
\vec{\alpha} &\sim MVN(0, 10\mathbf{I}_4) \\
\vec{\psi} &\sim N(0, 10\mathbf{I}_{K_{max}}) \\
\gamma_j &\sim \text{Uniform}[I_j],\ j = 1, \dots, K_{max}
\end{aligned}
$$

Prior specification

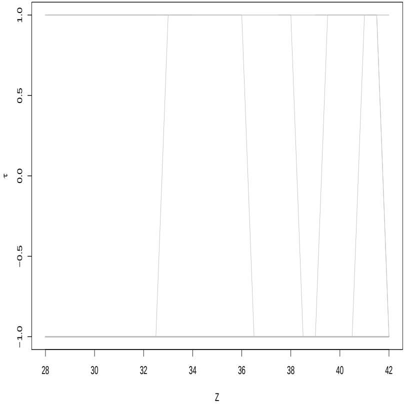


Figure: 500 curves generated from the prior spline model with $K_{max} = 4$.

Prior specification

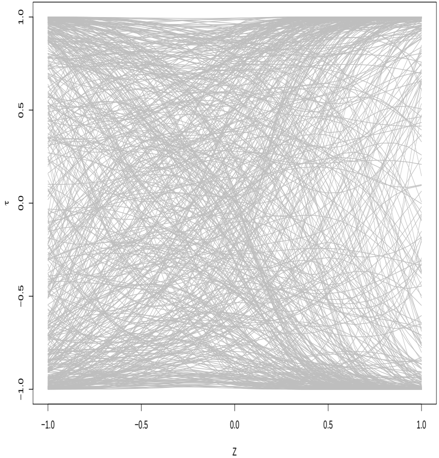


Figure: 500 curves generated from the prior spline model with $K_{max} = 4$. Covariate is standardized.

Multiple covariates

- Numerical problems when the number of covariates is large.
- Additive model approach

$$\eta(z_1, \ldots, z_r) = \sum_{j=1}^{r} \eta_j(z_j)$$

- Additivity is not preserved when changing dependence measure.
- When r is large we may want to use simpler models (piecewise constant).

Intermezzo

- So far:
  - Motivation for CC models
  - Continuous Response Case
  - Calibration function model
  - Multiple covariates.
- Next:
  - Mixed Response Case
  - Model selection issues
  - Simulation results.

Dependence models for CC's in Nonlinear Regression Models

- Measured outcome consists of a binary, Q, and a continuous r.v. W
- Assume a logistic regression model for the binary response.
- For subject j ($j = 1, \ldots, n$) we observe:

$Q_j \sim Bernoulli\left(\frac{\exp(\beta_1 X_j)}{1+\exp(\beta_1 X_j)}\right)$ the binary observed outcome

$W_j \sim N(\beta_2 X_j, \sigma^2)$: the continuous observed outcome

X_j : observed covariate

- Conditional copula dependence model for $Q, W|X$.

Dependence models for CC's in Nonlinear Regression Models (cont'd)

- Estimation for probit/logistic model involves the use of a latent normal/logistic random variable - the binary response variable is equal to the sign of the latent variable.
- Consider $Q_j = \mathbf{1}\{Y_j > 0\}$ where Y_j is a latent logistic random variable with density $f_L(y|A) = \frac{\exp(A-y)}{(1+\exp(A-y))^2}$.
- The latent variable Y is introduced to facilitate computation (for EM, DA).
- Sometimes Y offers a deeper interpretation of the model.
- Marginalization preserves the copula model for $(Y, W)|X$.

The Statistical Model

- The contribution of the jth sample to the observed-data likelihood

$$
\begin{aligned}
\Pr(Q_j = a, W_j | X_j, \omega) &= \frac{\phi\{(W_j - X_j\beta_2)/\sigma_2\}}{\sigma_2} \times \\
\times \quad & \left[c^{(0,1)} \left\{ \frac{\exp(aX_j\beta_1)}{1+\exp(X_j\beta_1)}, \Phi\left(\frac{W_j - X_j\beta_2)}{\sigma_2}\right) \middle| \theta(X_j) \right\} \right]^{1-a} \\
\times \quad & \left[1 - c^{(0,1)} \left\{ \frac{\exp\{(1-a)X_j\beta_1\}}{1+\exp(X_j\beta_1)}, \Phi\left(\frac{W_j - X_j\beta_2}{\sigma_2}\right) \middle| \theta(X_j) \right\} \right]^{a},
\end{aligned}
$$

where $a \in \{0, 1\}$ and ω represents the vector of all the parameters involved in the model.

The Statistical Model

- We assume that the dependence between the latent variable Y and W is characterized by the same conditional copula $C\{\cdot,\cdot|\theta(X)\}$.
- The contribution of the jth sample to the *complete data likelihood* would be (if the Y's were observed)

$$\begin{aligned} f(Y_j, W_j|X_j,\omega) &= f_L\left(Y_j|X_j\beta_1\right)\frac{1}{\sigma_2}\phi\left(\frac{W_j - X_j\beta_2}{\sigma_2}\right) \\ &\times \; c^{(1,1)}\left\{F_L\left(Y_j|X_j\beta_1\right), \Phi\left(\frac{W_j - X_j\beta_2}{\sigma_2}\right)\Bigg|\theta(X_j)\right\} \end{aligned}$$

- $\Pr(Q_j = 0, W_j|X_j,\omega) = \int\limits_{-\infty}^{0} f(Y_j, W_j|X_j,\omega)dY_j$

Remarks

- The usual dependence measures can be used in the bivariate continuous and mixed outcome models.
- If at least one variable is discrete, the dependence parameters are functions of all the parameters in the model,
- Conditional on X, the population Kendall's tau is

$$\begin{aligned}\tau(\omega|X) &= 4\mathrm{E}\{H(Q,W|X)|X\} - 1 = \frac{3 + 2\exp(\beta_1 X) + 3\exp(2\beta_1 X)}{\{1+\exp(\beta_1 X)\}^2} \\ &- 4\int_{\mathbf{R}} \left[\frac{1}{\sigma_2}\phi\left(\frac{w - X\beta_2}{\sigma_2}\right) \times \right. \\ &\times \left. C\left\{\frac{1}{1+\exp(X\beta_1)}, \Phi\left(\frac{w - X\beta_2}{\sigma_2}\right) \middle| \theta(X)\right\}\right] \mathrm{d}w,\end{aligned}$$

where $H(\cdot,\cdot|X)$ is the conditional joint cdf of (Q, W) given X.

Computation Algorithm

- The posterior distribution π is not analytically tractable.
- The sampling scheme requires to alternate between:
  i) sampling the latent variables from their conditional distribution and
  ii) sampling from the conditional posterior distribution of each parameter (given the complete data).

Computation Algorithm (cont'd)

- For most components (Y's, β's, $\log(\sigma)$, α's and ψ's) we use Metropolis-Hastings updates:

Step I Sample a proposal $\tilde{\omega} \sim q(\cdot|\omega_t, s_\omega)$

Step II $\omega_{t+1} = \tilde{\omega}$ with probability $\min\{1, \frac{\pi(\tilde{\omega}|EE)q(\omega_t|\tilde{\omega},s_\omega)}{\pi(\omega_t|EE)q(\tilde{\omega}|\omega_t,s_\omega)}\}$; otherwise $\omega_{t+1} = \omega_t$.

- Choice of s_ω is important and tuning can be time-consuming. We use adaptive MCMC to tune the s_ω's "on the go" .

Two Selection Problems

- Selection of the copula family among a set of candidates.
- Determine whether a parametric calibration function (esp. constant) is suitable.
- Deviance Information Criterion (DIC - Spiegelhalter et al, JRSSB '02).
- Cross-validated marginal likelihood criterion (Geisser and Eddy, JASA '79).

Selection via DIC

- The DIC is defined as

$$DIC(\mathcal{M}) = 2E[\Delta(\omega)|\mathcal{D}_{obs}] - \Delta(E[\omega|\mathcal{D}_{obs}]) \qquad (1)$$

  where the model deviance $\Delta(\omega) = -2\ln p(\mathcal{D}_{obs}|\omega, \mathcal{M})$
- All the required expectations in (1) can be computed using Monte Carlo samples.
- Models with the lowest DIC value are preferred.

A CV Marginal Likelihood Criterion

- The criterion computed under model $\mathcal{M}$ is

$$H(\mathcal{M}) = \sum_{i=1}^{n} \log p(Q_i, W_i|\mathcal{D}_{obs,-i}, \mathcal{M})$$

- $$E[f(Q_i, W_i|\vec{\zeta}, \vec{\omega})^{-1}] = \frac{1}{p(\mathcal{D}_{obs})} \int \frac{f(\mathcal{D}_{obs}|\vec{\zeta}, \vec{\omega})p(\vec{\zeta}, \vec{\omega})}{f(Q_i, W_i|\vec{\zeta}, \vec{\omega})} d\vec{\zeta} d\vec{\omega}$$

$$= \frac{1}{p(\mathcal{D}_{obs})} \int f(\mathcal{D}_{obs,-i}|\vec{\zeta}, \vec{\omega})p(\vec{\zeta}, \vec{\omega}) d\vec{\zeta} d\vec{\omega} = \frac{p(\mathcal{D}_{obs,-i})}{p(\mathcal{D}_{obs})}$$

$$= \frac{1}{p(Q_i, W_i|\mathcal{D}_{obs,-i})}$$

- $p(Q_i, W_i|\mathcal{D}_{obs,-i}, \mathcal{M}) \approx \left[\frac{1}{M} \sum_{m=1}^{M} \frac{1}{f(Q_i, W_i|\vec{\zeta}^{(m)}, \vec{\omega}^{(m)})}\right]^{-1}$,

Simulations

- We performed a large number of simulations to study:
  - the spline model's flexibility in capturing non-linear trends in the calibration function
  - the power to select the correct copula model and to determine the form of the calibration function.
  - the use of additive models for two covariates.
  - variable selection using two criteria.

Simulations

- We have generated data under the Clayton copula family for two sample sizes, $n = 150, 450$. Three dependence patterns are created using the following calibration functions:

$$\begin{aligned} &\text{C1: } \eta(z) = \ln(3); \\ &\text{C2: } \eta(z) = \ln\{0.07z^6 - 0.37(z+1)(z-0.5) + 0.3\}; \\ &\text{C3: } \eta(z) = \ln\{4.5 - 1.5\sin(\pi z)\}. \end{aligned}$$

- The data is analyzed using three copula families: Clayton, Frank and Gumbel with the corresponding link functions $g_C(x) = \ln(1 + x)$, $g_F(x) = x$ and $g_G(x) = \ln(x - 1)$, respectively.

Simulation Results

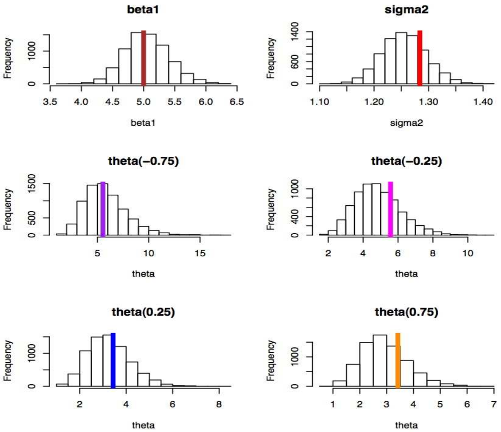
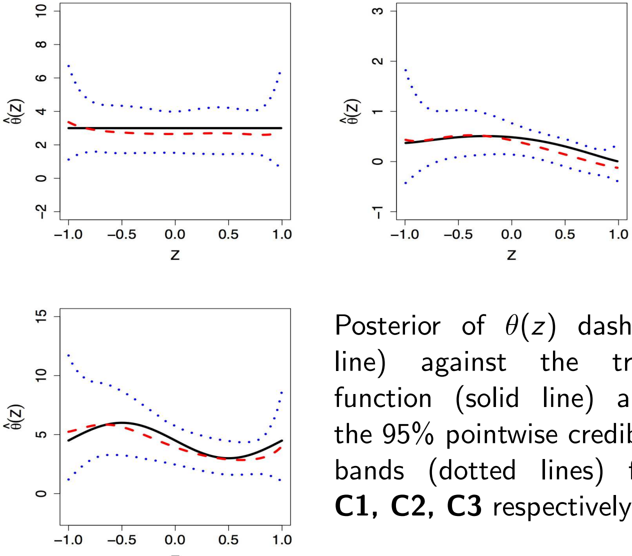


Figure: Histograms of posterior samples obtained from all the parallel MCMC chains when data is generated using a Clayton copula under scenario **C3**. The solid line marks the true parameter value.

Simulation Results



Posterior of $\theta(z)$ dashed line) against the true function (solid line) and the 95% pointwise credible bands (dotted lines) for **C1, C2, C3** respectively.

Estimation Results

Copula	$n = 150$ IBias^2	IVar	IMSE	$n = 450$ IBias^2	IVar	IMSE
	Scenario C1					
Clayton	**0.91**	**4.56**	**5.47**	**0.19**	**1.56**	**1.75**
Frank	23.84	2.90	26.74	28.08	1.94	30.02
Gumbel	5.36	11.62	16.98	4.80	4.18	8.98
	Scenario C2					
Clayton	**0.78**	**6.46**	**7.24**	**0.1**	**2.32**	**3.43**
Frank	4.07	5.94	10.01	3.65	1.98	5.63
Gumbel	18.70	16.64	35.34	17.35	14.61	31.96
	Scenario C3					
Clayton	**0.53**	**4.02**	**4.55**	**0.08**	**1.23**	**1.31**
Frank	21.00	2.25	23.25	22.24	1.43	23.67
Gumbel	2.67	10.88	13.55	5.83	3.43	9.26

Copula Selection

Copula Selection

Percentage of selecting
the Clayton family

$\Downarrow$

Copula Selection

Percentage of selecting the Clayton family

$\Downarrow$

Criteria	n	Frank	Gumbel
Scenario C1			
DIC	150	87%	70%
	450	98%	99%
CVML	150	62%	67%
	450	100%	91%
Scenario C2			
DIC	150	100%	92%
	450	98%	99%
CVML	150	99%	100%
	450	100%	99%
Scenario C3			
DIC	150	94%	80%
	450	100%	98%
CVML	150	86%	60%
	450	100%	97%

Copula Selection

Percentage of selecting
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DIC	150	87%	70%
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Scenario C3			
DIC	150	94%	80%
	450	100%	98%
CVML	150	86%	60%
	450	100%	97%

Copula Selection

Calibration Selection

Percentage of selecting the Clayton family

⇓

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DIC	150	87%	70%
	450	98%	99%
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	450	100%	91%
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DIC	150	100%	92%
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CVML	150	99%	100%
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Copula Selection

Percentage of selecting
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Scenario C3			
DIC	150	94%	80%
	450	100%	98%
CVML	150	86%	60%
	450	100%	97%

Calibration Selection

Percentage of selecting
the calibration function

$\Downarrow$

Copula Selection

Percentage of selecting
the Clayton family

⇓

Criteria	n	Frank	Gumbel
Scenario C1			
DIC	150	87%	70%
	450	98%	99%
CVML	150	62%	67%
	450	100%	91%
Scenario C2			
DIC	150	100%	92%
	450	98%	99%
CVML	150	99%	100%
	450	100%	99%
Scenario C3			
DIC	150	94%	80%
	450	100%	98%
CVML	150	86%	60%
	450	100%	97%

Calibration Selection

Percentage of selecting
the calibration function

⇓

Criteria	n	Spline Calibration	Constant Calibration
Scenario C1			
DIC	150	65%	**35%**
	450	49%	**51%**
CVML	150	66%	**34%**
	450	35%	**65%**
Scenario C2			
DIC	150	**93%**	7%
	450	**87%**	13%
CVML	150	**96%**	4%
	450	**100%**	0%
Scenario C3			
DIC	150	**78%**	22%
	450	**75%**	25%
CVML	150	**66%**	34%
	450	**96%**	4%

Copula Selection

Percentage of selecting
the Clayton family

$\Downarrow$

Criteria	n	Frank	Gumbel
Scenario C1			
DIC	150	87%	70%
	450	98%	99%
CVML	150	62%	67%
	450	100%	91%
Scenario C2			
DIC	150	100%	92%
	450	98%	99%
CVML	150	99%	100%
	450	100%	99%
Scenario C3			
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	450	100%	98%
CVML	150	86%	60%
	450	100%	97%

Calibration Selection

Percentage of selecting
the calibration function

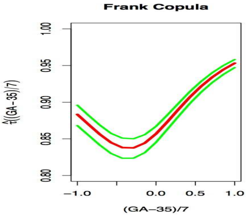
$\Downarrow$

Criteria	n	Spline Calibration	Constant Calibration
Scenario C1			
DIC	150	65%	**35%**
	450	49%	**51%**
CVML	150	66%	**34%**
	450	35%	**65%**
Scenario C2			
DIC	150	**93%**	7%
	450	**87%**	13%
CVML	150	**96%**	4%
	450	**100%**	0%
Scenario C3			
DIC	150	**78%**	22%
	450	**75%**	25%
CVML	150	**66%**	34%
	450	**96%**	4%

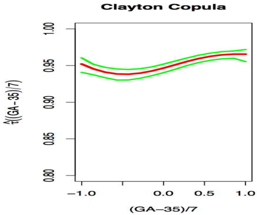
Twin Data Example

- Twin live births in which both babies survived in the first year of life with mothers of age between 18 and 40.
- Study the dependence between the birth weights of twins, BW_1 and BW_2.
- The gestational age, GA, is an important factor for prenatal growth and is therefore chosen as the covariate. We consider a random sample of 30 twin live births for each gestational age (in weeks) between 28 to 45.

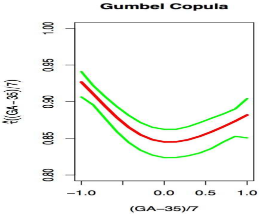
Twin Data Example



DIC=10449



DIC=14810

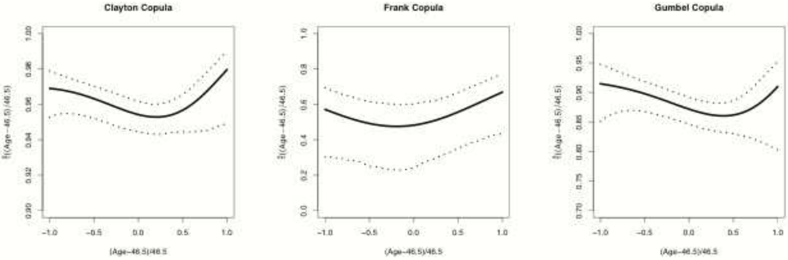


DIC=6.97×10^7

Example: Burn Injury

- $n = 981$ burn injury cases
- Occurrence of death $\Rightarrow Q = 1$ for death and $Q = 0$ for survival
- Total burn area $\Rightarrow W = \log(\text{burn area} + 1)$
- Patient's age
- How age effects the dependence between the severity of burn injury and the probability of death?

Burn Injury



Criteria	n	Clayton	Frank	Gumbel
DIC	981	6865.483	7082.946	6844.854
CVML	981	-3432.229	-3540.972	-3422.06

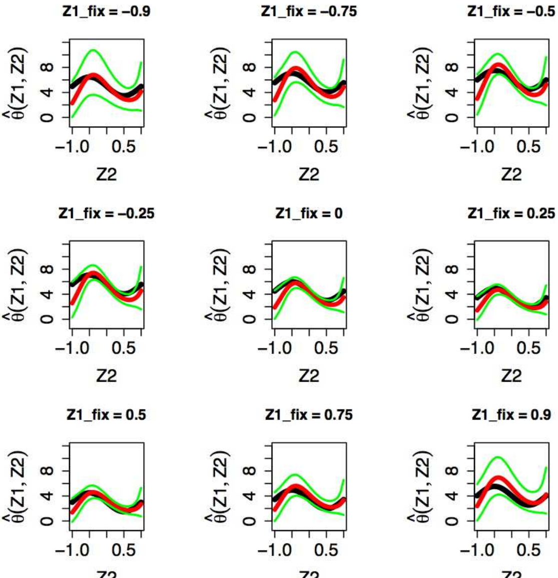
$\Uparrow$

chosen

Additive Model

- Z_1 & Z_2 are iid $N(0, 0.35^2)$.
- $\eta(z_1, z_2) = 4.5 - 1.5\sin(z_1\pi) - 1.5\sin(z_2\pi)$

Additive Model



Variable Selection

Variable Selection

- Data were generated under Clayton family; $n = 450$

Variable Selection

- Data were generated under Clayton family; $n = 450$
- X_1 & $X_2 \sim \mathcal{N}(0, 0.35^2)$

Variable Selection

- Data were generated under Clayton family; $n = 450$
- X_1 & $X_2 \sim \mathcal{N}(0, 0.35^2)$
- **C3** : $\eta(z) = \ln[4.5 - 1.5 \sin(\pi z)]$
- **C4** : $\eta(z_1, z_2) = \ln[4.5 - 1.5 \sin(\pi z_1) - 1.5 \sin(\pi z_2)]$

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Criteria	n	1-Covariate	2-Covariate
Scenario C3			
DIC	450	**96%**	4%
CVML	450	**96%**	4%
Scenario C4			
DIC	450	0%	**100%**
CVML	450	2%	**98%**

Conclusions

- Bayesian analysis for conditional copula models with bivariate responses $\rightarrow$ ... multivariate responses?
- Copula selection via DIC ($\checkmark$) and LP ($\checkmark$).
- Validation of a constant calibration function for continuous ($\checkmark$) and mixed ($\times$) responses.
- Additive models for multivariate predictors are promising.
- CC models offer a gateway into joint modelling of high dimensional data using low-dimensional copulas; the performance of this approach remains untested.

References

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