Bayesian Inference for Conditional Copulas using Gaussian Process Single Index Models

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Conditional Copula

- Consider a random sample \( \{x_i \in \mathbb{R}^d, y_1i \in \mathbb{R}, y_2i \in \mathbb{R}\}_{1 \leq i \leq n} \) and suppose \( F_X(y_1) \) and \( G_X(y_2) \) are the unknown marginal conditional cdf’s.

- The bivariate conditional copula (CC) of \( (Y_1, Y_2)|X = x \), is the conditional joint distribution function of \( U = F_X(Y_1) \) and \( V = G_X(Y_2) \) given \( X = x \) (Patton, Int’l Econ. Rev. ’06)

\[
H_x(t, s) = C_x(F_X(t), G_X(s))
\]

- The parametric bivariate CC model assumes there is a parametric family \( \mathcal{C} = \{ C_\theta : \theta \in \Theta \} \) s.t.

\[
C_x(F_X(t), G_X(s)) = C_{\theta(x)}(F_X(Y_1), G_X(Y_2)).
\]

- The simplifying assumption:

\[
C_x(F_X(y_1), G_X(y_2)) = C(F_X(y_1), G_X(y_2)).
\]
Why CC?

- We are interested in understanding the covariate effect on the dependence pattern between responses.

- Joint models for multivariate data: if $U_1, U_2, U_3 \sim \text{Uniform}(0, 1)$ then the joint pdf

$$c(u_1, u_2, u_3) = c_{12}(u_1, u_2)c_{23}(u_2, u_3)c_{\theta(u_2)}(F(u_1|u_2), G(u_3|u_2)).$$

- Regression-based prediction: if

$$h_x(y_1, y_2) = f_x(y_1)g_x(y_2)c_{\theta(x)}(F_x(y_1), G_x(y_2)),$$

then

$$h_x(y_1|y_2) = f_x(y_1)c_{\theta(x)}(F_x(y_1), G_x(y_2)).$$
Why CC? - Model misspecification effects

- Marginals
  - $f_1(x) = 0.6 \sin(5x_1) - 0.9 \sin(2x_2)$
  - $f_2(x) = 0.6 \sin(3x_1 + 5x_2)$
  - $\sigma_1 = \sigma_2 = 0.2, \ X_1 \perp X_2.$
- Copula: $\tau(x) = 0.71$
- Model:
  - Fit nonparametric model for marginals and CC with only $x_1$. 

$kendalls\ tau$
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![Graph showing marginal and copula distributions]

- Kendall's $\tau$ values are shown for different values of $x_1$. 

- The copula $\tau(x)$ is plotted against $x_1$. 

- The marginal distributions for $f_1(x)$ and $f_2(x)$ are also depicted.
Gaussian Process Prior

- GP prior for smooth $f$ without specifying the form of $f$.
- For $x \in [-5, 5]^n$, consider $f \sim N_n(0, K(x, x))$ where $K_{ij}(x, x) = k(x_i, x_j)$ and $f_i = f(x_i)$
- $\text{Cov}(f(x_i), f(x_j)) = k(x_i, x_j) = \exp\{-0.5 \times \frac{|x_i - x_j|^2}{L}\}$.

$L = 1$

$L = 5$

- Random functions $f$ generated from a GP prior when $n = 100$
Gaussian Process Estimation

- Observe \( \{y_i : 1 \leq i \leq n\} \) noisy realizations of \( f(x_i)_{i=1,n} \),
  \[ y_i = f(x_i) + \epsilon_i, \; \epsilon_i \sim N(0, \sigma^2). \]
- When interested in predicting \( f^* = (f(x_j^*))_{j=1,q} \) use
  \[
  \begin{pmatrix}
  y \\
  f^*
  \end{pmatrix}
  \sim N_{n+q}
  \begin{pmatrix}
  0, & \begin{bmatrix}
  K(x, x) + \sigma^2 I_n & K(x, x^*) \\
  K(x, x^*) & K(x^*, x^*)
  \end{bmatrix}
  \end{pmatrix}
  \]
- The conditional distribution of \( f^* \) is Gaussian with
  \[
  \mathbb{E}(f^*|y) = K(x^*, x) \left( K(x, x) + \sigma^2 I_q \right)^{-1} y
  \]
  \[
  \mathbb{V}(f^*|y) = K(x^*, x^*) - K(x^*, x) \left( K(x, x) + \sigma^2 I_q \right)^{-1} K(x, x^*)
  \]
  expensive for large \( n \)
Sparse GP-SIM

- When \( n \) is large the computation effort is prohibitive so we adopt a sparse GP approach (Snelson & Ghahramani 2005; Quiñonero-Candela & Rasmussen 2005)
- The information about \( f \) in the data is funnelled using a smaller sample of size \( m \ll n \) of inducing (or latent) variables \( \tilde{x}_g, 1 \leq g \leq m \).
- We consider the SIM model (Choi et al. 2011; Gramacy & Lian 2012)
  \[ f(X) = f(\beta^T X). \]
- GP-SIM model is invariant to nonlinear one-to-one transformations \( \tau(\theta) \).
\textbf{Proof of concept}

\textbf{Sc1} \quad f_1(x) = 0.6 \sin(5x_1) - 0.9 \sin(2x_2), \\
\quad f_2(x) = 0.6 \sin(3x_1 + 5x_2), \\
\quad \tau(x) = 0.7 + 0.15 \sin(15x^T \beta) \\
\quad \beta = (1, 3)^T / \sqrt{10}, \quad \sigma_1 = \sigma_2 = 0.2 \quad n = 400

\begin{center}
\begin{tabular}{l||c|c|c|c|c|c|c|c|c|c}
 & \text{Clayton} & & & \text{Frank} & & & \text{Gaussian} & & \text{Clayton SA} \\
\hline
\text{Scenario} & \sqrt{\text{IBias}}^2 & \sqrt{\text{IVar}} & \sqrt{\text{IMSE}} & \sqrt{\text{IBias}}^2 & \sqrt{\text{IVar}} & \sqrt{\text{IMSE}} & \sqrt{\text{IBias}}^2 & \sqrt{\text{IVar}} & \sqrt{\text{IMSE}} & \sqrt{\text{IBias}}^2 & \sqrt{\text{IVar}} & \sqrt{\text{IMSE}} \\
\hline
\text{Sc1} & 0.0231 & 0.0531 & 0.0579 & 0.1264 & 0.0322 & 0.1304 & 0.1434 & 0.0557 & 0.1539 & 0.0416 & 0.0579 & 0.0713
\end{tabular}
\end{center}

Integrated error for the estimator of $\tau(x)$. 
Prediction performance

- If $y_i | x \sim N(\mu_i(x), \sigma_i^2)$, $i = 1, 2$ then

$$E_x[Y_1 | Y_2 = y_2] = \mu_1(x) + \sigma_1 \int_0^1 \Phi^{-1}(z)c_{\theta(x)} \left(z, \Phi \left(\frac{y_2 - \mu_2(x)}{\sigma_2}\right)\right) \, dz.$$
Model Selection Problems

- Choice of copula family.
- Choice of calibration
  - Simplifying Assumption or not?
- Covariate selection.
CV Marginal Likelihood (CVML)

- Calculates the average (over parameter values) prediction potential for model $\mathcal{M}$ via

$$CVML(\mathcal{M}) = \sum_{i=1}^{n} \log (P(Y_{1i}, Y_{2i}|D_{-i}, \mathcal{M})),$$

- $D_{-i}$ is the data set from which the $i$th observation has been removed.
CV Marginal Likelihood (CVML)

- Estimate CVML using

\[
E_\pi \left[ P(Y_{1i}, Y_{2i}|\omega, \mathcal{M})^{-1} \right] = P(Y_{1i}, Y_{2i}|\mathcal{D}_{-i}, \mathcal{M})^{-1}
\]

where \(\omega\) represents the vector of all the parameters and latent variables in the model.

- Numerically

\[
\text{CVML} = \sum_{i=1}^{n} \log \left\{ \frac{1}{M} \sum_{t=1}^{M} \frac{1}{\sigma_1^{(t)}} \phi \left( \frac{y_{1i} - f_{1i}^{(t)}}{\sigma_1^{(t)}} \right) \frac{1}{\sigma_2^{(t)}} \phi \left( \frac{y_{2i} - f_{2i}^{(t)}}{\sigma_2^{(t)}} \right) \times \right. \\
\left. \times \ c_{\theta_i^{(t)}} \left[ \Phi \left( \frac{y_{1i} - f_{1i}^{(t)}}{\sigma_1^{(t)}} \right), \Phi \left( \frac{y_{2i} - f_{2i}^{(t)}}{\sigma_2^{(t)}} \right) \right] \right\}.
\]

- Add scenario S2 where SA is true:

\[
\begin{align*}
  f_1(x) &= 0.6 \sin(5x_1) - 0.9 \sin(2x_2) \\
  f_2(x) &= 0.6 \sin(3x_1 + 5x_2) \\
  \tau(x) &= 0.5 \\
  \sigma_1 &= \sigma_2 = 0.2
\end{align*}
\]
### Calibration Selection - Results

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Clayton CVML</th>
<th>Clayton CCVML</th>
<th>Frank CVML</th>
<th>Frank CCVML</th>
<th>Gaussian CVML</th>
<th>Gaussian CCVML</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sc2 (SA is true)</td>
<td>58%</td>
<td>62%</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
</tr>
</tbody>
</table>
A permutation based diagnostic for SA

- Randomly partition the data into a training set $\mathcal{D}_* = \{y_{1i}, y_{2i}, x_i\}_{i=1,...,n_1}$ (66% of sample) and a test set $\mathcal{D}^* = \{y_{1i}^*, y_{2i}^*, x_i^*\}_{i=1,...,n_2}$ (34% of sample).
- Fit marginal models using GP and a nonconstant calibration on $\mathcal{D}$. 
A permutation based diagnostic for SA

- Use training data to fit the calibration curve
- Use the test data to verify support for SA.
- Permutation influences only the copula factor.
A permutation based diagnostic for SA

- Consider $J$ permutations of $\{1 \ldots n_2\}$ which we denote as $\lambda_1, \ldots, \lambda_J : \{1, \ldots, n_2\} \rightarrow \{1, \ldots, n_2\}$
- Compute $J$ permuted CVMLs as:

$$CVML_j = \sum_{i=1}^{n_2} \log \left\{ \frac{1}{M} \sum_{t=1}^{M} \frac{1}{\sigma_1(t)} \phi \left( \frac{y_{1i}^* - f_{1i}^*(t)}{\sigma_1(t)} \right) \frac{1}{\sigma_2(t)} \phi \left( \frac{y_{2i}^* - f_{2i}^*(t)}{\sigma_2(t)} \right) \times \frac{1}{c_{\theta^*(t)}^{\lambda_j(i)}} \left[ \Phi \left( \frac{y_{1i}^* - f_{1i}^*(t)}{\sigma_1(t)} \right), \Phi \left( \frac{y_{2i}^* - f_{2i}^*(t)}{\sigma_2(t)} \right) \right] \right\}.$$ 

- If calibration is constant then $CVML_{\text{obs}}$ and $CVML_j$ should be similar
- Define the evidence

$$EV = 2 \times \min \left\{ \sum_{j=1}^{J} \mathbf{1}_{\{CVML_{\text{obs}} < CVML_j \}} \frac{1}{J}, \sum_{j=1}^{J} \mathbf{1}_{\{CVML_{\text{obs}} > CVML_j \}} \frac{1}{J} \right\}.$$ (1)
A permutation based diagnostic for SA

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Perm CVML</th>
<th>Perm CCVML</th>
<th>CVML</th>
<th>CCVML</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sc1</td>
<td>98%</td>
<td>96%</td>
<td>94%</td>
<td>94%</td>
</tr>
<tr>
<td>Sc2</td>
<td>92%</td>
<td>90%</td>
<td>58%</td>
<td>62%</td>
</tr>
<tr>
<td>Sc3</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
</tr>
</tbody>
</table>

Papers available at
http://www.utstat.toronto.edu/craiu/Papers/index.html