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CHOICE OF PARAMETRIC FAMILIES OF COPULAS

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Abstract

Copulas have evolved into a popular tool for modeling dependence in a large number of statistical models. Choosing a copula from an ever increasing set of possibilities presents difficulties that are well recognized in the literature. In this paper we investigate via simulation the effect of copula misspecification on various quantities of interest in the model such as conditional means and conditional variances. We also investigate methods to select among a number of candidate families of parametric copulas using nonparametric kernel smoothing and various distances between distributions. Both the Kullback-Leibler divergence and the Hellinger distance perform very well in this setting.

1. Introduction

Copulas have evolved into a widely used tool for modeling dependence for a large spectrum of statistical problems (e.g., Oakes [34-36]; Shih and Louis [40]; Hougaard [24]; Wang and Ding [46]; Escarela and Carrière

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[14]; Breymann et al. [4]; Patton [37]). The term *copula* was first introduced by Sklar [41] following some initial ideas by Höffding [23]. The crux of the method is the ability to flexibly “couple” fixed marginal continuous distributions into a multivariate distribution via a copula function. There has been a considerable effort in designing new families of copulas such as the Archimedean copulas (Genest and MacKay [18]), copulas with polynomial sections (Quesada-Molina and Rodríguez-Lallena [38]; Nelsen et al. [33]). There exists a vast literature on connections between dependence concepts and various families of copulas (Joe [26]; Nelsen [32]).

Given a data set and a particular setup in which the data has been collected, one needs to decide on a copula for the model considered and an estimation procedure for its parameters. The selection of an appropriate family of copulas is a notoriously difficult problem as was discussed by Durrleman et al. [13] in financial modeling context. Kim et al. [28] discuss the impact of marginal model misspecification on the estimation of the copula’s parameters. Recent progress has been achieved by Fermanian [15] who proposed a goodness-of-fit test for copulas and by Wang and Wells [47] and Genest et al. [21] who propose selection methods based on the Kendall process introduced by Genest and Rivest [19]. Techniques for estimation of copula parameters were initiated by Deheuvels [10] and followed by maximum likelihood methods, inference function for margins (Joe [27]), semiparametric methods (Genest et al. [20]; Tsukahara [42]), nonparametric methods (Capéraà et al. [6]), Bayesian methods (Romeo et al. [39]) and minimum distance density estimation (Biau and Wegkamp [3]).

It is well known that estimation for the marginal distribution parameters are unaffected by the choice of the copula function used for modeling dependence. However, in some practical instances such as those discussed by Chaieb et al. [7] or Chen and Fan [8] we are interested in quantities which are related to the dependence structure between the variables under consideration. For instance, in the case of two dependent random variables X and Y whose joint distribution is specified using a copula model we may be interested in estimating the conditional means $\mathrm{E}[X \mid Y = y]$ and the conditional variances $\mathrm{Var}[X \mid Y = y]$ for different values y. The conditional mean and conditional variance depend on the copula function used to model the dependence between X and Y. More precisely, if X, Y are continuous random variables with distribution functions (df) F_X and, respectively, F_Y we specify the joint df using the copula $C : [0, 1] \times [0, 1] \rightarrow [0, 1]$ such that

$$F_{XY}(F_X^{-1}(u), F_Y^{-1}(v)) = \mathrm{P}(X \leq F_X^{-1}(u), Y \leq F_Y^{-1}(v)) = C(u, v). \tag{1}$$

Equation (1) illustrates the way in which the copula function "bridges" the marginal and the joint df's. The existence of such a map C is guaranteed by Sklar's Theorem (Sklar [41]). The uniqueness of C, once we fix F_X, F_Y and F_{XY}, is ensured as long as the random variables are continuous. In many instances we have a good idea about the marginal df's and little or no idea about the joint df. In the present paper we begin by investigating via simulations the effect of the copula's choice on conditional means and variances. We also propose a method for choosing among a set of parametric families of copulas using nonparametric smoothing combined with Monte Carlo-based estimation of distances between distributions.

Section 2 contains a description of the simulation models and the empirical findings related to the effect of copula misspecification. In Section 3 we present the copula selection procedure based on Kullback-Leibler (Kullback and Leibler [29]) and Hellinger distances. Further work is discussed in Section 4.

2. Specification of the Simulation Model

We consider a bivariate random vector (X, Y) with marginal df's F_X and F_Y. We assume two parametric forms for the marginals, those given by the exponential and the Weibull distributions. These distributions are commonly used in survival data modeling, an area where copula models are also frequently used. The parameters of the marginals df's are not identical. As mentioned in the previous section, the purpose of this study is to measure the effect of copula misspecification. We generate the data $\{(x_i, y_i) : 1 \leq i \leq n\}$ using the Clayton copula (Clayton [9])

$$C_\theta(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}. \tag{2}$$

However, we analyze the data produced using Frank's copula (Frank [16])

$$C_\alpha(u, v) = -\frac{1}{\alpha}\log\left[1 + \frac{(e^{-\alpha u} - 1)(e^{-\alpha v} - 1)}{e^{-\alpha} - 1}\right]. \quad (3)$$

Both C_θ and C_α are members of the Archimedean family of copulas and both are intensively used to model dependencies in survival data studies.

2.1. Data simulation

Since both distributions have invertible df's, we can work directly on $[0, 1]^2$. In other words, we need to sample independently $v_i \sim$ Uniform $(0, 1)$ and $u_i \mid v_i$ will need to be sampled from a conditional distribution such that (U_i, V_i) has df given by (2).

The *copula density* corresponding to (2) is

$$c_\theta(u, v) \propto (1 + \theta)u_1^{-\theta-1}u_2^{-\theta-1}(u_1^{-\theta} + u_2^{-\theta} - 1)^{-\frac{1+2\theta}{\theta}}. \quad (4)$$

In order to sample from $p(u \mid v) \propto c_\theta(u, v)$ we use a Metropolis-Hastings algorithm (Hastings [22]) with Uniform $(0, 1)$ proposal distribution. The sampler performs very well having an acceptance rate uniformly greater than 40%. For each $1 \leq i \leq n$ we sample $M = 2000$ samples from $p(u \mid v_i)$ and retain the last sample point obtained, $u_i = u_M$. The final samples are obtained using $x_i = F_X^{-1}(u_i)$, $1 \leq i \leq n$. In Figure 1 we overlap the histogram of a sample of size $n = 500$ from the conditional distribution of $X \mid Y$ obtained using the Markov chain Monte Carlo (MCMC) method described with the true density which in this case is Exp(2). This approach also allows us to compute, for any value y_0, the true $\mathrm{E}[X \mid Y = y_0]$ and $\mathrm{Var}(X \mid Y = y_0)$.

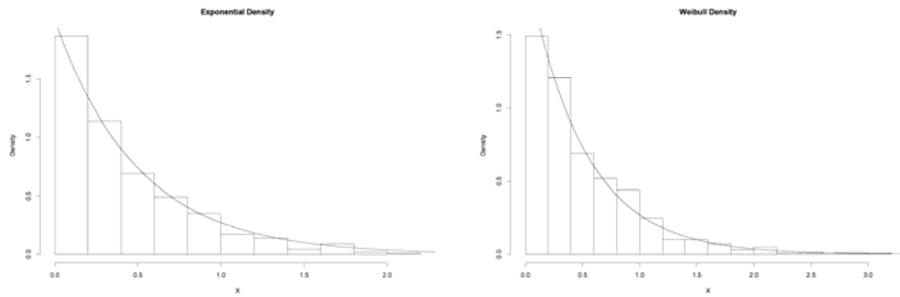

Figure 1. Sample of size 500 from the conditional distribution of $X \mid Y$. The continuous line is the true marginal density: Exponential (left) and Weibull (right).

2.2. Copula-based conditional inference

Given the data $\{(x_i, y_i) : 1 \leq i \leq n\}$ we need to estimate the parameter α of the copula given in (3). We will assume that the marginal parameters have been estimated perfectly from the two samples, in other words we will make partial use of the perfect knowledge we have as data creators. However, we do this because we want to assess directly and as accurately as possible the effect of copula misspecification. Since the marginal distributions are assumed known and the df's are invertible we can, once again work within the unit square, $[0, 1]^2$.

The copula density for Frank's copula is

$$c_\alpha(u, v) \propto \frac{\alpha e^{-\alpha(u+v)}(1 - e^{-\alpha})}{(e^{-\alpha} + e^{-\alpha(u+v)} - e^{-\alpha u} - e^{-\alpha v})^2}. \tag{5}$$

The maximum likelihood estimator $\hat{\alpha}$ corresponding to the density (5) is obtained via a simple grid search. Other numerical methods can also be used (e.g., Newton-Raphson) but the grid search performed quite well in our analysis. Using $\hat{\alpha}$ we compute the best prediction for X given $Y = y_0$, i.e., $\mathrm{E}[X \mid Y = y_0]$ along with $\mathrm{Var}(X \mid Y = y_0)$. The calculation is completed in a manner similar to the one described in the simulation section by running a Metropolis-Hastings MCMC algorithm designed to sample from the conditional distribution $p(u \mid v)$ induced by (5). In order to increase the accuracy of our results we consider samples of size 5,000 from the densities of interest.

2.3. Effect of copula misspecification: simulation results

We performed the study described above with $\theta = 3$, $F_X = \text{Exp}(2)$ and $F_Y = \text{Exp}(1)$ as well as $F_X = F_Y = \text{Weibull}(1, 2)$ and we obtained the results as described in Table 1. Table 2 summarizes results in the case in which we generated data with sensibly larger values of $\theta = 12$, the parameter in the Clayton copula. A slightly bigger similarity between Clayton and Frank copulas for larger values of θ and α results in slightly more accurate predictions for the conditional mean in the case in which the marginals are Weibull distributed. Predictions for the conditional variance remain significantly biased. In all situations presented as well as in other settings we explored and which are not reported here the effect of copula misspecification is significant and can have a large impact on inference.

3. Copula Selection

As we have seen in the previous section, the choice of the parametric family of copulas most suitable for the data at hand is important especially if the inference is concerned with parameters depending on the copula's association parameters. In this section we propose an approach based on nonparametric kernel smoothing and distributional distances.

3.1. General description of the method

In the following we will assume that the samples are bivariate with support on the unit square $(0, 1)^2$. This assumption relies on the fact that the marginal df's are known or can be estimated accurately and are invertible (or can be inverted numerically). In principle the approach proposed here can be used in a more general setting where the samples are in R^2 without substantial difference. However, our simulations are done in the $[0, 1]^2$ case.

Given a general distributional distance $\mathcal{D}$ and a sample of size n on the unit square $[0, 1]^2$ we would like to be able to select from a finite set of k parametric families of copulas one that is best representing the type of correlation exhibited by the data. Without loss of generality we assume that $k = 2.$ The method will choose the family which yields the smallest distance between the nonparametric estimate of the "true" density to the best fit obtained within each family. To be more precise, by best fit we mean the member of the family indexed by the maximum likelihood estimator of the copula parameter for that family. In general, the distance cannot be computed in closed form so we need to rely on Monte Carlo approximations. To this end it is important to be able to generate samples from the candidate copulas and to be able to estimate the nonparametric density constructed from the original samples at these simulated values. In general, both of these requirements can be met in practice.

One can see that central to our approach is: (a) the distributional distance chosen to measure the discrepancy between the nonparametric approximation to the true density and the fitted copulas as well as (b) the method used to construct the nonparametric approximation to the true density. In the remaining of this section we discuss the nonparametric kernel smoothing approximation and a couple of distributional distances which were used in simulations.

3.2. Kernel density estimation

In a practical setting in which we assume that marginal df's are known (or can be estimated accurately) suppose that we have available n samples from the copula density, $c^*(u, v).$ We propose to use as an approximation of the true copula density, $c^*,$ a bivariate kernel density estimate $\hat{c}^*(u, v)$ of $c^*(u, v)$ constructed using the available samples $\{(u_i, v_i) \in [0, 1]^2 : 1 \le i \le n\}.$

The nonparametric kernel density estimation of c^* is obtained using unconstrained and data-driven bandwidth matrices obtained as discussed in Wand and Jones [43, 45], Duong and Hazelton [11, 12] and as implemented in the R software package *ks*. If $X_1, ..., X_n$ are drawn from a d-dimensional density f, then the kernel density is defined by

$$\hat{f}(x;\ H) = n^{-1}\sum_{i=1}^{n} K_H(x - X_i),$$

where $x = (x_1, \ldots, x_d)^T$ and $X_i = (X_{i1}, \ldots, X_{id})$, $i = 1, 2, \ldots, n$. The kernel K is a symmetric probability density function, the bandwidth matrix H is symmetric and positive definite and $K_H(x) = |H|^{-1/2} K(H^{-1/2}x)$. The choice of K is not crucial and we use the standard d-dimensional Gaussian density throughout the paper. Unlike the choice of K, the choice of H is central to the performance of the smoother $\hat{f}$. In the simulation we allow H to be non-diagonal acknowledging that there is large probability mass oriented away from the co-ordinate directions (Duong and Hazelton [11, 12]). The bandwidth is selected via least squares cross validation, for details see Wand and Jones [45].

3.3. Distributional distance and its estimation

Definition 3.1. Given two densities f and g the *Kullback-Leibler* (*KL*) *distance* is defined as

$$\mathrm{KL}(f,\ g) = \int \log(f(x)/g(x))f(x)dx. \tag{6}$$

Hölder's inequality guarantees that $\mathrm{KL}(f,\ g) \geq 0$ and is zero if and only if $f = g$. KL has been used intensively in statistics for model selection purposes (Burnham and Anderson [5]) most famously in the well-known Akaike Information Criterion (AIC) introduced by Akaike [1]. In the derivation of the AIC one makes the assumption that there exists a true model belonging to the family of models under consideration. Another widely used distance between distributions is the Hellinger distance.

Definition 3.2. Given two densities f and g the *Hellinger* (*HE*) *distance* is defined as

$$\mathrm{HE}(f,\ g) = \left\{\int [\sqrt{f(x)} - \sqrt{g(x)}]^2 dx\right\}^{\frac{1}{2}}. \tag{7}$$

The form (7) can be rewritten as

$$\mathrm{HE}^2(f,\ g) = \int f(x)\left[1 - \frac{\sqrt{g(x)}}{\sqrt{f(x)}}\right]^2 dx. \tag{8}$$

To simplify the discussion suppose we are interested in choosing between two families of copula densities $\mathcal{A} = \{c_\alpha : \alpha \in A\}$ and $\mathcal{B} = \{c_\beta : \beta \in B\}$. Using the available samples $\{(u_i,\ v_i) \in [0,\ 1]^2 : 1 \le i \le n\}$ we estimate $\hat{\alpha}$ and $\hat{\beta}$ via maximum likelihood methods. Estimate $\mathrm{KL}(c_{\hat{\alpha}},\ \hat{c}^*)$ using a sample $\{(\tilde{u}_i,\ \tilde{v}_i) : 1 \le i \le m\}$ drawn from $c_{\hat{\alpha}}$ and using

$$\widehat{\mathrm{KL}}(c_{\hat{\alpha}},\ \hat{c}^*) = \frac{1}{m}\sum_{i=1}^{m} c_{\hat{\alpha}}(\tilde{u}_i,\ \tilde{v}_i)[\log(c_{\hat{\alpha}}(\tilde{u}_i,\ \tilde{v}_i)) - \log(\hat{c}^*(\tilde{u}_i,\ \tilde{v}_i))]. \tag{9}$$

Similarly we will estimate $\mathrm{KL}(c_{\hat{\beta}},\ \hat{c}^*)$ using

$$\widehat{\mathrm{KL}}(c_{\hat{\beta}},\ \hat{c}^*) = \frac{1}{m}\sum_{i=1}^{m} c_{\hat{\beta}}(\tilde{u}_i,\ \tilde{v}_i)[\log(c_{\hat{\beta}}(\tilde{u}_i,\ \tilde{v}_i)) - \log(\hat{c}^*(\tilde{u}_i,\ \tilde{v}_i))]. \tag{10}$$

One can notice that while the samples used in (9) and (10) are generated from $c_{\hat{\alpha}}$ and $c_{\hat{\beta}}$, respectively, the distances depend on $\log(\hat{c}^*(\tilde{u}_i,\ \tilde{v}_i))$. Given the same samples as used in (9) and (10) we estimate

$$\widehat{\mathrm{HE}^2}(c_{\hat{\alpha}},\ \hat{c}^*) = \frac{1}{m}\sum_{i=1}^{m}\left[1 - \frac{\sqrt{\hat{c}^*(\tilde{u}_i,\ \tilde{v}_i)}}{\sqrt{c_{\hat{\alpha}}(\tilde{u}_i,\ \tilde{v}_i)}}\right]^2 \tag{11}$$

and

$$\widehat{\mathrm{HE}^2}(c_{\hat{\beta}},\ \hat{c}^*) = \frac{1}{m}\sum_{i=1}^{m}\left[1 - \frac{\sqrt{\hat{c}^*(\tilde{u}_i,\ \tilde{v}_i)}}{\sqrt{c_{\hat{\beta}}(\tilde{u}_i,\ \tilde{v}_i)}}\right]^2. \tag{12}$$

3.4. Performance of the copula selection procedure

In Table 3 we report some of the simulation results. Each cell reports the correct number of selections, for different combinations of the value for the true copula family, θ, sample size n and distributional distance, KL or HE. One can see that the criterions based on the KL distance and on the Hellinger distance are performing very well. It is well known that the Hellinger distance is sensitive to the way mass is allocated within the support of the distribution (Gelman and Meng [17]) and we suspect this is the reason for the good performance exhibited here.

4. Conclusions and Further Work

We have discussed the effect of copula misspecification on conditional inference. Simulations show that the effect can be significant and a careful choice of the copula family is needed. In the context of selection between a number of parametric families of copulas we propose a selection procedure that relies on the Kullback-Leibler divergence or Hellinger distance and a nonparametric kernel density estimate reconstructed from the available data.

Further work is needed in order to compare the behavior of the current selection procedure with other existent copula selection methods as well as study the performance of the method proposed here in the case of multivariate copulas.

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Table 1. Comparison between the true and estimated predictors for X and true and estimated conditional variances of X for various values of $Y = y_0$. Each cell contains the bias estimated over 100 replicates and between brackets is the Monte Carlo standard error for the bias estimate

Clayton's $\theta = 3$; $F_X = \text{Exp}(2)$, $F_Y = \text{Exp}(1)$						
y_0	0.5	0.7	1.0	1.5	2.0	2.5
Bias(E$[X\|Y = y_0]$)	-0.067 (0.009)	-0.084 (0.011)	-0.072 (0.014)	-0.003 (0.022)	0.075 (0.031)	0.140 (0.037)
Bias(Var$[X\|Y = y_0]$)	0.142 (0.026)	0.215 (0.032)	0.364 (0.043)	0.646 (0.080)	0.869 (0.124)	1.041 (0.147)
Clayton's $\theta = 3$; $F_X = F_Y = \text{Weibull}(1, 2)$						
y_0	0.5	0.7	1.0	1.5	2.0	2.5
Bias(E$[X\|Y = y_0]$)	-0.052 (0.042)	-0.166 (0.045)	-0.285 (0.048)	-0.357 (0.051)	-0.301 (0.057)	-0.170 (0.071)
Bias(Var$[X\|Y = y_0]$)	-0.061(0.018)	-0.294 (0.118)	-0.647 (0.209)	-1.036 (0.279)	-1.139 (0.343)	-1.030 (0.400)

Table 2. Comparison between the true and estimated predictors for X and true and estimated conditional variances of X for various values of $Y = y_0$.

Clayton's $\theta = 12$; $F_X = \text{Exp}(2)$, $F_Y = \text{Exp}(1)$						
y_0	0.5	0.7	1.0	1.5	2.0	2.5
$\text{Bias}(\text{E}[X\|Y = y_0])$	0.019 (0.006)	0.020 (0.007)	0.015 (0.010)	-0.016 (0.018)	-0.056 (0.028)	-0.086 (0.041)
$\text{Bias}(\text{Var}[X\|Y = y_0])$	0.119 (0.014)	0.193 (0.019)	0.338 (0.032)	0.604 (0.073)	0.824 (0.111)	1.003 (0.166)
Clayton's $\theta = 12$; $F_X = F_Y = \text{Weibull}(1, 2)$						
y_0	0.5	0.7	1.0	1.5	2.0	2.5
$\text{Bias}(\text{E}[X\|Y = y_0])$	0.011 (0.012)	0.002(0.013)	-0.008(0.016)	-0.035 (0.023)	-0.072 (0.032)	-0.118 (0.047)
$\text{Bias}(\text{Var}[X\|Y = y_0])$	0.056 (0.006)	0.067 (0.008)	0.076 (0.014)	0.050 (0.043)	-0.061 (0.096)	-0.294 (0.295)

Table 3. Performance of the copula selection procedure proposed with the distributional distance being Kullback-Leibler (KL) or Hellinger (HE). The numbers in each cell represent the percentage of correct selections. The data is generated using a Clayton copula and the candidate families are Clayton and Frank

Method\\n	50	100	300	500
Clayton's $\theta = 3$				
KL	100	100	100	100
HE	99	99	100	100
Clayton's $\theta = 8$				
KL	100	100	100	100
HE	100	100	100	100
Clayton's $\theta = 12$				
KL	100	100	100	100
HE	100	100	100	100

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