STA 107F
Midterm Exam #1
October 9, 2002
3:00 P.M.

SOLUTIONS

a) There are 5 pages including this page. Please check to see if you have all the pages.

b) A calculator is permitted, but not needed. If the final answer to a question is numeric, then you may give that answer using a fractional or decimal representation.

c) There are 40 marks available.

d) You may use the following facts without proving them. Note that $A \cap B$ and AB both mean the same thing: "A intersect B".

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

$$P(A) = 1 - P(A^c)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(\emptyset) = 0$$

1. **(10 marks total)** Let S be a sample space of outcomes. Let P be a probability function. Let A and B be events in S.

a) **(4 marks)** Prove that $P(A \cap B) \geq 1 - P(A^c) - P(B^c)$.

$$\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\
P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\
&\geq P(A) + P(B) - 1 \\
&= (1 - P(A^c)) + (1 - P(B^c)) - 1 \\
&= 1 - P(A^c) - P(B^c)
\end{aligned}$$

Other solutions are possible.

b) **(6 marks)** Suppose that $P(A) = 1/6$ and $P(A \cup B) = 1/4$. For each of the following three conditions calculate $P(B)$.

i. **(2 marks)** A and B are disjoint.

$$\begin{aligned}
P(A \cup B) &= P(A) + P(B) \\
1/4 &= 1/6 + P(B) \\
P(B) &= 1/12
\end{aligned}$$

ii. **(2 marks)** $P(A|B) = P(B|A)$

$$\begin{aligned}
\frac{P(A \cap B)}{P(B)} &= \frac{P(B \cap A)}{P(A)} \\
P(B) &= P(A) \\
&= 1/6
\end{aligned}$$

iii. **(2 marks)** A and B are independent.

$$\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\
P(A \cup B) &= P(A) + P(B) - P(A)P(B) \\
P(A \cup B) &= P(A) + P(B)[1 - P(A)] \\
1/4 &= 1/6 + P(B)[5/6] \\
P(B) &= 1/10
\end{aligned}$$

2. **(10 marks total)** (Read the following if you not familiar with the standard six-sided die: *A die (plural: dice) is a cube with sides numbered 1,2,3,4,5, and 6. When a die is thrown, assume that each side is equally likely to face up when the die comes to rest.*)

Suppose that 4 fair dice are thrown. Assume that the outcomes of the 4 dice are independent.

a) **(5 marks)** What is the probability that each of the 4 dice displays a number greater than or equal to 4, i.e. a 4, 5, or a 6?

Denote this event by A, and let S be the sample space. $|S| = 6^4 = 1296$ and $|A| = 3^4$. So $P(A) = 3^4/6^4 = 2^{-4} = 1/16 = 0.0625$.

b) **(5 marks)** What is the probability that the 4 dice each show a different number?

Denote this event by B. Then $|B| = 6 \cdot 5 \cdot 4 \cdot 3 = 360$. So $P(B) = 360/6^4 = 360/1296 = 0.278$.

3. **(10 marks total)** You ask your neighbor to feed your fish while you are on vacation. Without food, it will die with probability 0.6; with food it will die anyway with probability 0.05, because fish like to break your heart. You are 80% certain that your neighbor will remember to feed your fish.

Let A be the event your neighbor feeds the fish. Let B be the event your fish dies.

a) **(5 marks)** What is the probability that the fish will be alive when you return?

$$\begin{aligned} P(B) &= P(B|A)P(A) + P(B|A^c)P(A^c) \\ &= 0.05 \cdot 0.8 + 0.6 \cdot 0.2 \\ &= 0.16 \end{aligned}$$

So $P(B^c) = 1 - 0.16 = 0.84$

Some students will have sketched a tree diagram, and read the answer from it. That is an acceptable answer too.

b) **(5 marks)** If it is dead, what is the probability that your neighbor forgot to feed it?

$$\begin{aligned} P(A^c|B) &= \frac{P(A^c \cap B)}{P(B)} \\ &= \frac{P(B|A^c)P(A^c)}{P(B|A^c)P(A^c) + P(B|A)P(A)} \\ &= \frac{0.6 \cdot 0.2}{0.16} \\ &= 0.75 \end{aligned}$$

Some students will have sketched a tree diagram, and read the answer from it. That is an acceptable answer too.

4. **(10 marks total)** How many different arrangements of the letters

$$\text{A B C D E F}$$

are there such that:

a) **(5 marks)** A and B are next to each other? (e.g. F C **A B** E D)

Denote this event by M $|M| = 5! \cdot 2 = 240$.

Other solutions possible (that end up with 240!)

If a student explicitly stated that they believed that A and B had to appear together *and in that order*, and came up with the answer 120, they should get no more than 4/5.

If a student just gave the answer 120, then there's no way of knowing if they knew what they were doing but misread the question, or they understood the question and did it wrong. They should get no more than 3/5.

b) **(5 marks)** A is after B and B is after C? (e.g. **C** D **B A** F E, and note that it doesn't matter if any of A, B, and C are adjacent or not.)

Denote this event by N. Now, imagine finding all *distinct* arrangements of the letters:

$$\text{x x x D E F}$$

For each such arrangement, C, B, and A can be placed in that order where each "x" appears. The number of distinct arrangements is:

$$\begin{aligned} |N| &= {}_6P_3 \\ &= \frac{{}_6P_6}{{}_3P_3} \\ &= \frac{6!}{3!} \\ &= 6 \cdot 5 \cdot 4 \\ &= 120 \end{aligned}$$

Other solutions possible (that end up with 120!).