Joint Distribution - Review

• Let $X : S \to \{a_1, ..., a_n, ...\}$ and $Y : S \to \{b_1, ..., b_m, ...\}$ be two discrete random variables. **The joint probability mass function** of (X, Y) is $p : \{a_1, ..., a_n, ...\} \times \{b_1, ..., b_m, ...\} \to [0, 1]$

$$p(a_i, b_j) = P(X = a_i, Y = b_j)$$

• The **marginal distribution** of X has as probability mass function

$$p_X(a_i) = \sum_{k=1}^{\infty} p(a_i, b_k)$$

Similarly, the **marginal distribution** of Y has as probability mass function

$$p_Y(b_j) = \sum_{k=1}^{\infty} p(a_k, b_j).$$

Expectation - Review

Consider $g : \{a_1, ..., a_n, ...\} \times \{b_1, ..., b_m, ...\} \to R$:

- $E[g(X,Y)] = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} p(a_i, b_j) g(a_i, b_j)$
- $E[\alpha X + \beta Y] = \alpha E[X] + \beta E[y]$
- $E[\sum_{i=1}^{n} \alpha_i X_i] = \sum_{i=1}^{n} \alpha_i E[X_i]$.

Independence

- X and Y are independent random variables if and only if $p(a_i, b_j) = p_X(a_i)p_Y(b_j)$.

- If X and Y are independent $E[XY] = E[X]E[Y]$.

- If X **and** Y **are independent**

$$Var(\alpha X + \beta Y) = \alpha^2 Var(X) + \beta^2 Var(Y).$$

- If $X_1, \ldots, X_n$ **are independent**

$$Var(\sum_{i=1}^{n} \alpha_i X_i) = \sum_{i=1}^{n} \alpha_i^2 Var(X_i).$$

Example If $X \sim$ Binomial(n, p), then

$$Var(X) = np(1 - p).$$

Conditional Distribution

- The **conditional distribution** of X given that $Y = b_j$ is

$$P(X = a_i | Y = b_j) = p(a_i | b_j) = \frac{p(a_i, b_j)}{p_Y(b_j)}$$

for any value b_j such that $p_Y(b_j) > 0$.

- Note that $\sum_{i=1}^{\infty} p(a_i | b_j) = 1$.

- $E[g(X)|Y = b_j] = \sum_{i=1}^{\infty} p(a_i | b_j) g(a_i)$

- $E[g(Y)|X = a_i] = \sum_{j=1}^{\infty} p(b_j | a_i) g(b_j)$