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ON A CHARACTERIZATION QUESTION FOR SYMMETRIC RANDOM VARIABLES

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Abstract: If X_1, X_2 are independent with common density g symmetric about zero, then $P(X_1 + \alpha X_2 > 0) = \frac{1}{2}$ for all real α. We provide a counter example to show that the converse is false and thus settle a question posed by Burdick (1972).

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A substantial part of the theory of characterization problems is devoted to the deduction of properties of a summand from properties of the characteristic funtion of the sum (see e.g. Lukacs (1970) or Kagan, Linnik and Rao (1973) for numerous instances). Because it is so simple to characterize symmetric distributions as exactly those with real characteristic functions, it is intriguing to note that as soon as summand inference is involved subtleties can begin to arize. The problem studied in Burdick (1972) and pursued here is an instance of this phenomenon.

Suppose that X_1 and X_2 are independent random variables with common density $g(x)$. If $g(x)$ is symmetric about the origin then we may readily see that

$$P(X_1 + \alpha X_2 > 0) = \tfrac{1}{2} \tag{1}$$

for all real α. It is somewhat tempting to believe that characterizes symmetric densities. Burdick (1972) proved that if a fractional moment

$$\int |x|^{\varepsilon} g(x)\,\mathrm{d}x < \infty \tag{2}$$

exists for some $\varepsilon > 0$ than (1) implies $g(x) = g(-x)$ almost everywhere. Burdick then asked if the condition that a fractional moment exists could be omitted. In this note we construct an asymmetric g for which (1) holds and so provide a negative answer to Burdick's question. Our construction is based on a lemma of Freedman and Diaconis (1982).

We first state the following result.

Lemma. *Let X_1, X_2 be independent with a common continuous distribution function G. Then*

$$P(X_1 + \alpha X_2 \leqslant 0) = \tfrac{1}{2} \quad \textit{for all } \alpha \tag{3}$$

if and only if

$$\phi_1(t)\phi_2(t) = 0 \quad \textit{for all } t \tag{4}$$

where ϕ_1, and ϕ_2 are the Fourier–Stieltjes tranforms

$$\phi_1(t) = \int_{-\infty}^{\infty} \mathrm{e}^{itx}\,\mathrm{d}[G(\mathrm{e}^x) - G(-\mathrm{e}^x)]$$
$$= \int_{-\infty}^{\infty} y^{it}\,\mathrm{d}[G(y) - G(-y)] \tag{5}$$

and

$$\phi_2(t)=\int_{-\infty}^{\infty} e^{itx}\,d[G(e^{-x})+G(-e^{-x})]$$
$$=\int_{\infty}^{\infty} y^{-it}\,d[G(y)+G(-y)]. \tag{6}$$

Proof. By the convolution formula (e.g. Feller (1970, p. 144)) we have that (3) is equivalent to

$$\int_{-\infty}^{\infty} G\left(-\frac{x}{\alpha}\right) dG(x)=\tfrac{1}{2}, \quad \alpha\neq 0, \tag{7}$$

together with

$$G(0)=\tfrac{1}{2}. \tag{8}$$

Further (7) is equivalent to the pair of conditions

$$\int_{-\infty}^{\infty}\left[G\left(\frac{x}{\alpha}\right)-G\left(-\frac{x}{\alpha}\right)\right] dG(x)=0, \quad \alpha>0, \tag{9}$$

$$\int_{-\infty}^{\infty}\left[G\left(\frac{x}{\alpha}\right)+G\left(-\frac{x}{\alpha}\right)\right] dG(x)=1, \quad \alpha>0. \tag{10}$$

But actually (8) is a consequence of (10): just make the change of variables $x/\alpha \to y$ and take the limit $\alpha\to\infty$. Furthermore, (9) and (10) actually are equivalent to each other. To see this, first write (9) and (10), respectively, in the forms

$$\int_{0}^{\infty}\left[G\left(\frac{x}{\alpha}\right)-G\left(-\frac{x}{\alpha}\right)\right] d[G(x)+G(-x)]=0, \tag{11}$$

$$\int_{0}^{\infty}\left[G\left(\frac{x}{\alpha}\right)+G\left(-\frac{x}{\alpha}\right)\right] d[G(x)-G(-x)]=1 \tag{12}$$

and then apply intergration by parts to (12); make the change of variables $x/\alpha\to x$; finally the change of notation $\alpha\to 1/\alpha$. Hence, altogether, (3) and (11) are equivalent.

Next in (11) put $\alpha=e^{-t}$ and make the variable change $x\to e^{-y}$ to get

$$\int_{-\infty}^{\infty}[G(e^{t-y})-G(-e^{t-y})]$$
$$\cdot d[G(e^{-y})+G(-e^{-y})]=0 \quad \text{for all } t. \tag{13}$$

But this is the convolution of $G(e^{x})-G(-e^{-x})$ and $G(e^{-x})+G(-e^{-x})$ whose Fourier–Stieltjes transforms are ϕ_1 and ϕ_2. Consequently (13) is equivalent to (4).

Remark. The condition in the lemma that G be continuous may be omitted provided we replace (3) by

$$P(X_1+\alpha X_2<0)+\tfrac{1}{2}P(X_1+\alpha X_2=0)=\tfrac{1}{2}.$$

The same proof continues to hold except that instead of using right-continuous G we use the symmetric form $G(x)=\frac{1}{2}(G(x+)-G(x-))$ with appropriate conventions for the Steiltjes integral.

Turning now to the counterexample let G have denstiy g so that

$$\phi_1(t)=\int_{-\infty}^{\infty} e^{itx}u(x)\,dx$$

and

$$\phi_2(t)=\int_{-\infty}^{\infty} e^{itx}v(x)\,dx$$

where

$$u(x)=e^{x}(g(e^{x})+g(-e^{x}))$$

and

$$v(x)=e^{-x}(g(e^{-x})-g(-e^{-x})).$$

Observe that u and v here are in correspondence with the even and odd parts of g (but need not in themselves be even or odd.) Now, according to the proof of Lemma 3 of Freedman and Diaconis (1982), there exists a probability density of the form

$$f=c(f_1+\delta f_2)$$

having characteristic function

$$\hat{f}=c(\hat{f}_1+\delta\hat{f}_2)$$

such that $f_1\geqslant 0$, $\hat{f}_1$ is real and vanishes off $[-1, 1]$, and $\hat{f}_2$ is purely imaginary and vanishes off $[-3,-2]\cup[2,3]$.

The functions f_1 and f_2 can be selected so that $\delta\neq 0$, thus making sure that f is not symmetric. If we now set

$$u(x)=cf_1(x) \quad\text{and}\quad v(x)=c\delta f_2(-x),$$

the nonzero segments of the Fourier transforms of u and v will not overlap. On solving for g we obtain

$$g(u)=\frac{c}{2}|u|^{-1}\{f_1(\ln|u|)+(\operatorname{sgn} u)\delta f_2(\ln|u|)\}.$$

By our construction we see that g is a density which satisfies (4) and therefore (3). Because $\delta \neq 0$ we have that g is not symmetric, and thus the problem posed by Burdick (1972) is solved.

As a final observation, we note that by Burdick's theorem, g cannot possess a fractional moment. We can however show that g can posses a κ-th logarithmic moment for any particular value of κ. This follows from an immediate change of variables and from the fact that the construction of f_1 and f_2 permits them to have moments of arbitrarily large order.

We do not know if it is possible to construct a density satisfying (1) which posses logarithmic moments of all orders.

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