COMMUN. STATIST.-THEORY METH., 17(5), 1557-1564 (1988)

AN ASYMPTOTIC NEYMAN-PEARSON TYPE RESULT UNDER SYMMETRY CONSTRAINTS

Andrey Feuerverger and Sucharita Ghosh

Department of Statistics
University of Toronto
Toronto, Canada M5S 1A1

Key Words and Phrases: Neyman-Pearson lemma; likelihood ratio test; symmetry constraints; exact Bahadur slope; asymptotic relative efficiency.

ABSTRACT

The asymptotically optimal test statistics of the form $T_n = \frac{1}{n}\sum_{i=1}^{n} U(X_i)$ for independent identically distributed observations in the Neyman-Pearson context are derived under symmetry constraints on the function $U(\cdot)$, specifically for $U(\cdot)$ required to be symmetric or antisymmetric (i.e. an even or odd function). The calculations are based on the exact Bahadur slope and variational methods are required. Expressions for the asymptotic relative efficiency of these procedures are derived.

1. INTRODUCTION

Let $X_1, X_2, ..., X_n$ be identically and independently distributed random variables from density f. For the problem of testing H_o: $f = f_o$ versus H_1: $f = f_1$, the Neyman-Pearson optimal test statistic may be written in the form

$$T_n = \frac{1}{n}\sum_{i=1}^{n} U(X_i) = \int U(x)\,dF_n(x) \tag{1.1}$$

where

$$U(x) = \log \frac{f_1(x)}{f_o(x)} \,. \tag{1.2}$$

Our object here is to consider the situation where the function $U(\cdot)$ in (1.1) is subject to symmetry constraints, specifically, the cases where $U(\cdot)$ is required to be an even or an odd function; it turns out that these two cases are not of similar difficulty and in the odd case, only an asymptotic treatment is possible. Our results are given in the theorem and lemma of the following section.

As one example of how constraints such as those cited may arise, let $c_n(t) = \int e^{itx} dF_n(x)$ be the empirical characteristic function, and note that by Parseval's theorem T_n may be represented in the form $T_n = \int c_n(t)\,dW(t)$. In this context it may be of interest to determine the optimal real part and imaginary part based test statistics of the type $T_n^S = \int \mathrm{Re}\, c_n(t)\,dW(t)$ and $T_n^A = \int \mathrm{Im}\, c_n(t)\,dW(t)$, both of which are of the form (1.1) but with $U(\cdot)$ now being symmetric (even) and antisymmetric (odd) respectively.

In the context of a simple null hypothesis and simple alternative, asymptotic optimization may be suitably effected through the (exact) Bahadur slope. For some helpful discussion and references, see Feuerverger (1987).

2. THE RESULTS

Our principal result is the following:

Theorem: For testing $H_o: f = f_o$ versus $H_1: f = f_1$ the asymptotically $(n \to \infty)$ optimal constrained test statistic of the form (1.1) is given:

(i) in the symmetric case, but provided that $f_1(x) + f_1(-x)$ and $f_o(x) + f_o(-x)$ are not almost everywhere equal, by

$$U(x) = \log\left[\frac{f_1(x) + f_1(-x)}{f_o(x) + f_o(-x)}\right]; \tag{2.1}$$

(ii) in the antisymmetric case, but provided that $f_1(x) - f_o(x)$ is not an even function, by

$$U(x) = \pm \log R(x) \tag{2.2}$$

where

$$R(x) = \frac{c \cdot f_1^A(x) + \sqrt{(c \cdot f_1^A(x))^2 + 4 \cdot f_o(x)\, f_o(-x)}}{2 \cdot f_o(x)}, \tag{2.3}$$

$f_1^A(x) = f_1(x) - f_1(-x)$, and c is the unique positive root of

$$\int \sqrt{(f_1^A(x))^2 + 4 \cdot c^{-2} \cdot f_o(x) f_o(-x)}\, dx = 2. \tag{2.4}$$

When $f_1(x)$ is even, the solution may be taken as

$$U(x) = \pm \log\left[\frac{f_o(-x)}{f_o(x)}\right]. \tag{2.5}$$

Proof. We consider first the antisymmetric case (ii) where $U(\cdot)$ is constrained to be an odd function and write $U(\cdot)$ in the form

$$U(x) = V(x) - V(-x). \tag{2.6}$$

Then

$$T_n = \frac{1}{n}\sum_1^n \left[V(X_i) - V(-X_i)\right] \tag{2.7}$$

has, under H_o and H_1 respectively, expectations

$$E_o \equiv E_o T_n = \int [V(x) - V(-x)] f_o(x)\, dx \tag{2.8}$$

and

$$E_1 \equiv E_1 T_n = \int [V(x) - V(-x)]\, f_1(x)\, dx \tag{2.9}$$

and we may require $E_o T_n < E_1 T_n$ so that large values of T_n constitute evidence against H_o. Then by a theorem of Bahadur (see Serfling (1980), p.337) the exact slope of T_n is

$$\lim_{n \to \infty} \left[-\frac{2}{n} \log P_{H_o}[T_n \geq E_1] \right], \tag{2.10}$$

and by Chernoff's large-deviation theorem (ibid pp.326-328) this

$$= -2 \log \inf_z \int e^{z(V(x) - V(-x) - E_1)} f_o(x)\, dx. \tag{2.11}$$

We thus seek to minimize

$$\int e^{z(V(x) - V(-x) - E_1)}\, f_o(x)\, dx \tag{2.12}$$

subject to the additional constraint

$$\int e^{z(V(x) - V(-x) - E_1)} [V(x) - V(-x) - E_1]\, f_o(x)\, dx = 0 \tag{2.13}$$

obtained by differentiating in z in (2.11). There are now two cases which arise according as $f_1(\cdot)$ is or is not an even function. We consider first the case where $f_1(\cdot)$ is not even. In this case we may require that

$$E_1 = \int [V(x) - V(-x)]\, f_1(x)\, dx = 1 \tag{2.14}$$

thus fixing the scale of V. In the event that (2.14) violates the assumption $E_o < E_1$ we would take instead $E_1 = -1$ with only minor changes resulting.

We shall thus extremize (2.12) subject to the constraints (2.13) and (2.14). Let z, V(x) be the required solution and introduce variants δz and $\delta V(x)$. Then by (2.14) we must have

$$\int \left[\delta V(x) - \delta V(-x)\right] f_1(x)\, dx = 0. \tag{2.15}$$

Similarly (2.13) gives (ignoring second order terms)

$$\int e^{z(V^A(x) - 1)} \left[\delta z \left[V^A(x) - 1\right]^2 + \delta V^A(x) \left[z(V^A(x) - 1) + 1\right]\right] f_o(x)\, dx = 0 \tag{2.16}$$

where we have introduced the notations $V^A(x) = V(x) - V(-x)$ and $\delta V^A(x) = \delta V(x) - \delta V(-x)$. This relation is seen to give δz in terms of δV but plays no further role below. Next, introducing the variants in (2.12), ignoring second order terms, and applying the standard argument, we have that the first order term must be always zero since (2.12) is taken to be already at its maximum; using (2.13) the resulting relation may be written as

$$\int e^{z(V^A(x) - 1)}\, \delta V^A(x)\, f_o(x)\, dx = 0. \tag{2.17}$$

Rewriting (2.15) and (2.17) in the forms

$$\int \delta V(x) \left[f_1(x) - f_1(-x)\right] dx = 0 \tag{2.18}$$

and

$$\int \delta V(x) \left[f_o(x)\, e^{z(V^A(x) - 1)} - f_o(-x)\, e^{z(-V^A(x) - 1)}\right] dx = 0\,, \tag{2.19}$$

we see that $\left[f_o(x)\, e^{z(V^A(x)-1)} - f_o(-x)\, e^{z(-V^A(x)-1)}\right]$ must be orthogonal to every function orthogonal to $f_1{}^A(x) \equiv f_1(x) - f_1(-x)$ so that

$$f_o(x)\, e^{z(V^A(x)-1)} - f_o(-x) e^{z(-V^A(x)-1)} = a \cdot f_1{}^A(x) \tag{2.20}$$

where (a, z) must be chosen to satisfy (2.13) and (2.14).

Let now $c = a \cdot e^z$ so that (2.20) becomes

$$f_o(x) e^{zV^A(x)} - f_o(-x) e^{-zV^A(x)} = c \cdot f_1^A(x), \tag{2.21}$$

and observe next that this may be cast as a quadratic equation in the unknown $e^{zV^A(x)}$, which must be positive; this leads to the solution

$$V^A(x) = \frac{1}{z} \log \left[\frac{c \cdot f_1^A(x) + \sqrt{(c \cdot f_1^A(x))^2 + 4 \cdot f_o(x) f_o(-x)}}{2 \cdot f_o(x)} \right]. \tag{2.22}$$

Observe further that if $\int dx \left[V^A(x) - 1\right]$ is applied to (2.20) then in view of (2.13) we obtain

$$-\int e^{z(-V^A(x)-1)} \left[V^A(x) - 1\right] f_o(-x)\, dx = a \cdot \int \left[V^A(x) - 1\right] f_1^A(x)\, dx = 2 \cdot a \tag{2.23}$$

where we have used (2.14). This together with (2.13) establishes $-2 \log (a)$ as the maximum attainable slope. On the other hand, if (2.22) is substituted in (2.12) we obtain $-2 \log \frac{1}{2} e^{-z} \int \sqrt{(c \cdot f_1^A(x))^2 + 4 \cdot f_o(x) f_o(-x)}\, dx$ as an alternate expression for that slope and equating the two expressions leads to the equation (2.4) for the constant c. The factor of z in (2.22) has been omitted from the expression (2.2) without consequence.

In the case that $f_1(x)$ is even, $E_1 = 0$ and the constraint (2.14) must be replaced by $E_o = -1$. The result stated for this case follows by appropriate modifications to the argument.

Finally, the symmetric case (i) can be treated in the manner above; the calculations that result are somewhat simpler and lead to (2.1). Alternatively, the result for this case follows more easily as a simple consequence of the finite sample optimality result given in the lemma stated below. ■

In the case that $U(\cdot)$ is even, the test statistic is invariant as $(X_1,...,X_n)$ ranges over the 2^n possible sign changes of the X's; the maximal invariant in this case is just the vector of absolute values $(|X_1|, \ldots, |X_n|)$ and has density $\prod_{i=1}^{n} [f(x_i)+f(-x_i)]$. The Neyman-Pearson lemma then applies to give the following elementary result:

Lemma: For arbitrary, fixed sample size n, the optimal statistic of the form (1.1) for testing H_o versus H_1 based on a symmetric $U(\cdot)$ is given by (1.2).

The result for odd $U(\cdot)$ however appears not to have any finite sample interpretation or analogue.

Finally, the asymptotic relative efficiency of the even and odd based tests relative to the Neyman-Pearson test is of interest, and may be taken to be the ratio of the respective Bahadur slopes. Now the exact slope of the likelihood ratio test is well known to be twice the Kullback-Leibler information number so that we obtain immediately, for the even case, an efficiency of

$$e = \frac{\int_0^\infty (f_1(x) + f_1(-x)) \log \left[\frac{f_1(x) + f_1(-x)}{f_o(x) + f_o(-x)}\right] dx}{\int_{-\infty}^\infty f_1(x) \log \frac{f_1(x)}{f_o(x)}\, dx} . \tag{2.24}$$

To obtain an expression for the asymptotic efficiency in the odd case, we note that when $f_1(x)$ is not even, the slope is given by $-2\log a = 2(z - \log c)$. One expression for z may be obtained on substituting (2.22) in (2.14) and results in the representation

$$e = \frac{2\int_{-\infty}^{\infty} f_1(x)\log\left[\frac{f_1^A(x) + \sqrt{(f_1^A(x))^2 + 4\cdot c^{-2}\cdot f_o(x)f_o(-x)}}{2\cdot f_o(x)}\right]dx}{\int_{-\infty}^{\infty} f_1(x)\log\frac{f_1(x)}{f_o(x)}\,dx} \tag{2.25}$$

for the efficiency where c is the positive root of equation (2.4). However, when $f_1(x)$ is even, the corresponding calculations show that the asymptotic efficiency must be taken as

$$e = \frac{-2\log\int_{-\infty}^{\infty}\sqrt{f_o(x)f_o(-x)}\,dx}{\int_{-\infty}^{\infty} f_1(x)\log\frac{f_1(x)}{f_o(x)}\,dx}. \tag{2.26}$$

ACKNOWLEDGEMENTS

This work was supported in part by a grant from the National Sciences and Engineering Research Council of Canada.

BIBLIOGRAPHY

Feuerverger, A. (1987). "New derivations for the maximum likelihood estimator and the likelihood ratio test." *Communications in Statistics; Theory and Methods, 16*(5), 1241-1251.

Serfling, R.J. (1980). *Approximation Theorems in Statistics.* New York: J. Wiley and Sons.

Received by Editorial Board member October, 1987; Revised November, 1987.

Recommended by Edwin L. Crow, National Telecommunications and Information Administration, Boulder, Colorado.