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On the bootstrap saddlepoint approximations

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SUMMARY

We compare saddlepoint approximations to the exact distributions of a studentized mean and to its bootstrap approximation. We show that, on bounded sets, these empirical saddlepoint approximations achieve second order relative errors uniformly. We also consider the relative errors for larger deviations. It follows that the studentized-t bootstrap p-value and the coverage of the bootstrap confidence interval have second order relative errors.

Some key words: Bootstrap-t p-value and coverage; Empirical saddlepoint approximations.

1. INTRODUCTION

Let $X_1, \ldots, X_n$ be identically and independently distributed random variables having probability density $f(x)$ and distribution function $F(x)$. Let $x_1, \ldots, x_n$ be the observed values of $X_1, \ldots, X_n$, let $F_n(x)$ be the empirical distribution function and let $X_1^*, \ldots, X_n^*$ be identically and independently distributed random variables with distribution function $F_n(x)$.

Define the distribution functions

$$Q(a_1) = \text{pr}\,\{n^{\frac{1}{2}}(\bar{X} - \mu)/S \geqslant a_1 \,|\, F\},$$

$$Q^*(a_1) = \text{pr}\,\{n^{\frac{1}{2}}(\bar{X}^* - \bar{x})/S^* \geqslant a_1 \,|\, F_n\},$$

where

$$\bar{X} = n^{-1} \sum_{i=1}^{n} X_i, \quad \bar{X}^* = n^{-1} \sum_{i=1}^{n} X_i^*, \quad S^2 = n^{-1} \sum_{i=1}^{n} (X_i - \bar{X})^2, \quad S^{*2} = n^{-1} \sum_{i=1}^{n} (X_i^* - \bar{X}^*)^2$$

and $\mu = E(X_1)$. Since we are dealing with studentized variates we may, without loss of generality, assume that $E(X_1) = 0$ and $E(X_1^2) = 1$. Assume that in some open rectangle in (t, u) containing the origin

$$K(t, u) = \log E\{e^{tX_1 + u(X_1^2 - 1)}\}$$

exists. Then, in the Appendix, we prove the following.

THEOREM 1. *Under the above conditions, for* $a_1 = o(n^{1/3})$,

$$Q(a_1) = Q^*(a_1)\left\{1 + O_p\left(\frac{a_1^3}{n}\right)\right\}.$$

In § 2, we use this result to show that a bootstrap approximation to the p-value of a test based on the studentized mean has relative error of $O_p(t^3/n)$, where t is the observed value of the studentized mean under the null hypothesis, and that under contiguous alternatives this is $O_p(n^{-1})$. Also, we show that the relative error of the coverage of the bootstrap-t $(1-2\alpha)$-confidence interval is $O(z_\alpha^3/n)$, where z_α is the $1-\alpha$ quantile of the standardized normal distribution. These methods extend, under the stronger conditions assumed here, the results of Hall (1988), where it is shown that the absolute error of coverage is $O(n^{-1})$. As Hall (1990) pointed out, simulation studies show that the bootstrap usually performs better than such Edgeworth expansion arguments suggest and he gave, for standardized means, a comparison of the relative error of the bootstrap and the Edgeworth approximation formula.

The proof of Theorem 1, given in the Appendix, consists of first obtaining saddlepoint approximations for $Q(a_1)$ and $Q^*(a_1)$ of the form proposed by Barndorff-Nielsen (1986, 1991) and shown, by Jensen (1992), to be equivalent to the form given by Lugannani & Rice (1980) and extended to studentized means by Daniels & Young (1991). These approximations are known to have relative error of $O(n^{-1})$, uniformly for a_1 in the range $(0, O(n^{\frac{1}{2}}))$. The theorem follows by comparing these saddlepoint approximations using Taylor expansions, noting that the coefficients are simple functions of low order cumulants and so differ by $O_p(n^{-\frac{1}{2}})$.

We can obtain similar but simpler results for standardized means. Further, we can obtain results comparing the saddlepoint approximations to the densities of the standardized and studentized means and their empirical counterparts, the extensions of the empirical saddlepoint of Feuerverger (1989) to the cases of standardized and studentized means. Saddlepoint approximation methods have been considered as an alternative to computationally demanding bootstrap and resampling procedures; see for example Davison & Hinkley (1988), Wang (1989, 1990) and Robinson (1982). For work of related interest, see also DiCiccio, Martin & Young (1992).

2. SOME BOOTSTRAP APPLICATIONS

2·1. *Application to the bootstrap p-value*

Suppose we want to test $H_0: \mu = \mu_0$ against $H_1: \mu > \mu_0$, where μ is the mean of F. If we choose $T = n^{\frac{1}{2}}(\bar{X} - \mu_0)/S$ as our test statistic, then the observed significance level is $p = \text{pr}(T \geqslant t \mid \mu_0)$, where $t = n^{\frac{1}{2}}(\bar{x} - \mu_0)/s$. Define $T^* = n^{\frac{1}{2}}(\bar{X}^* - \bar{X})/S^*$, where $X_1^*, \ldots, X_n^*$ is a random sample with replacement from F_n. Then the bootstrap approximation to the significance level p is given by $p^* = \text{pr}(T^* \geqslant t \mid \mu_0)$. From Theorem 1, we see that

$$p = p^*\left\{1 + O_p\left(\frac{t^3}{n}\right)\right\}.$$

If $\mu - \mu_0 = O(1/n^{\frac{1}{2}})$ we will have $t = O_p(1)$, so that $p = p^*\{1 + O_p(n^{-1})\}$.

2·2. *Application to the bootstrap coverage error*

Hall (1988) showed that the studentized-t bootstrap method produces one-sided confidence limits which are second order accurate, and hence results in bootstrap coverage having error of $O(n^{-1})$. We will show that this is actually a relative error. To do this, we first need an expression for the bootstrap coverage. Thus define x_α^* as the solution of

$$\text{pr}^*\left(\frac{\bar{X}^* - \bar{x}}{S^* n^{-\frac{1}{2}}} \leqslant x_\alpha^*\right) = \alpha,$$

where pr* indicates the conditional probability given $\mathscr{X} = \{X_1, \ldots, X_n\}$. Note that x_α^* is to be regarded here as a random variable. Then the bootstrap coverage is given by

$$\pi^*(\alpha) \equiv E_{x_\alpha^*} \operatorname{pr}\left(\frac{\bar{X}-\mu}{Sn^{-\frac{1}{2}}} \leqslant x_\alpha^* \,|\, x_\alpha^*\right).$$

We may now establish the following.

THEOREM 2. *Under the conditions of Theorem* 1, *the bootstrap coverage probability satisfies*

$$\pi^*(\alpha) = \alpha\left\{1 + O\left(\frac{z_\alpha^3}{n}\right)\right\},$$

where $\Phi(z_\alpha) = 1-\alpha$.

Proof. From Theorem 1, for $x_\alpha^* = o_p(n^{1/3})$, we have

$$\frac{\operatorname{pr}\{n^{\frac{1}{2}}(\bar{X}-\mu)/S \leqslant x_\alpha^* \,|\, x_\alpha^*\}}{\operatorname{pr}\{n^{\frac{1}{2}}(\bar{X}^*-\bar{x})/S^* \leqslant x_\alpha^* \,|\, x_\alpha^*\}} = 1 + O_p\left(\frac{x_\alpha^{*3}}{n}\right);$$

that is

$$\operatorname{pr}\left(\frac{\bar{X}-\mu}{Sn^{-\frac{1}{2}}} \leqslant x_\alpha^* \,\middle|\, x_\alpha^*\right) = \alpha\left\{1 + O_p\left(\frac{x_\alpha^{*3}}{n}\right)\right\}.$$

Now letting $A \equiv (z_\alpha - \delta, z_\alpha + \delta)$ for a fixed δ, we can show, using a Cornish–Fisher expansion for x_α^* (Hall, 1988) and a Chebyshev bound, that

$$\operatorname{pr}(x_\alpha^* \in A) = 1 - O(n^{-1}).$$

Consequently

$$\begin{aligned}\pi^*(\alpha) &= E_{x_\alpha^*} \operatorname{pr}\left(\frac{\bar{X}-\mu}{Sn^{-\frac{1}{2}}} \leqslant x_\alpha^* \,\middle|\, x_\alpha^*\right)\\ &= \operatorname{pr}\left(\frac{\bar{X}-\mu}{Sn^{-\frac{1}{2}}} \leqslant x_\alpha^* \,\middle|\, x_\alpha^* \in A\right)\operatorname{pr}(x_\alpha^* \in A) + \operatorname{pr}\left(\frac{\bar{X}-\mu}{Sn^{-\frac{1}{2}}} \leqslant x_\alpha^* \,\middle|\, x_\alpha^* \in \bar{A}\right)\operatorname{pr}(x_\alpha^* \in \bar{A})\\ &= \alpha\left\{1 + O\left(\frac{z_\alpha^3}{n}\right)\right\}.\end{aligned}$$

□

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APPENDIX

Proof of Theorem 1

Daniels & Young (1991) obtain the tail area approximation to $Q(a_1)$:

$$\hat{Q}_n(a_1) = \int_{w_1}^{\infty} \left(\frac{n}{2\pi}\right)^{\frac{1}{2}} e^{-nw^2/2}\psi(w)\,dw,$$

where we have written $\psi(w)$ for their $\psi(a, b_0)$ of equation (4·12) by taking a as a function of w defined by inverting their equation (4·9), and w_1 is defined by replacing a by a_1 in thier equation (4·9). This is accurate to relative error of $O(n^{-1})$ for $a_1 = O(n^{\frac{1}{2}})$, or $O(n^{-3/2})$, for a_1 bounded. Integrating this using Temme's method (Temme, 1982) gives the approximation

$$\hat{Q}_n(a_1) = 1 - \Phi(w_1 n^{\frac{1}{2}}) - \frac{\Phi(w_1 n^{\frac{1}{2}})}{n^{\frac{1}{2}}} \left\{ \frac{1}{w_1} - \frac{\psi(w_1)}{w_1} \right\},$$

which is equivalent, to relative error of $O(n^{-1})$, to

$$\hat{Q}_n(a_1) = 1 - \Phi(\hat{w} n^{\frac{1}{2}})$$

as shown by Jensen (1992), where $\hat{w} = w_1 - \log \psi(w_1)/(n w_1)$. This last result can also be obtained directly as follows:

$$\begin{aligned}
\hat{Q}_n(a_1) &= \left(\frac{n}{2\pi}\right)^{\frac{1}{2}} \int_w^\infty \exp\left(-\frac{1}{2} n w^2\right) \psi(w)\, dw \\
&= \left(\frac{n}{2\pi}\right)^{\frac{1}{2}} \int_w^\infty \exp\left[-\frac{1}{2} n \left\{ w - \frac{\log \psi(w)}{nw} \right\}^2 \right] \exp\left\{ \frac{\log^2 \psi(w)}{2nw^2} \right\} dw \\
&= \left(\frac{n}{2\pi}\right)^{\frac{1}{2}} \int_{\hat{w}}^\infty \exp\left(-\frac{1}{2} n u^2\right) \frac{dw}{du}\, du \{1 + O(n^{-1})\} \\
&= (1 - \Phi\{\hat{w} n^{\frac{1}{2}}\}) \{1 + O(n^{-1})\}
\end{aligned}$$

since

$$\frac{du}{dw} = 1 + \frac{\log \psi(w)}{nw^2} - \frac{\psi'(w)}{\psi'(w) nw}.$$

In order to obtain the same approximation for $Q^*(a_1)$, we need to show that the tail probability obtained by integration of the formal density estimate is accurate to the stated order when the conditions for the existence of a density do not hold. Here, because of the continuity of the underlying distribution, an indirect Edgeworth approximation, with relative error $O(n^{-3/2})$ may be shown to hold. This is proved by B.-Y. Jing and J. Robinson, in an unpublished report, for studentized means, using similar conditions to those used for means by Robinson et al. (1990). Then we note that the integral of the formal density is equal to the indirect Edgeworth expansion to $O(n^{-3/2})$, since this holds in cases where the density exists, and so in other cases, as this equality is not dependent on the continuity assumptions, but purely on the form of the expansions. So the saddlepoint approximation for $Q^*(a_1)$ is

$$\hat{Q}_n^*(a_1) = \{1 - \Phi(\hat{w}^* n^{\frac{1}{2}})\} \{1 + O_p(n^{-1})\},$$

where $\hat{w}^* = w_1^* - \log \psi^*(w_1^*)/(n w_1^*)$. The only difference here is that w_1^* and $\psi^*(w_1^*)$ are defined in the same way as w_1 and $\psi(w_1)$ except that the empirical cumulant generating function is used. Expanding $w_1 = w(a_1)$ using a Taylor series, we can show that

$$w(a_1) = \frac{a_1}{n^{\frac{1}{2}}} + A_1 \frac{a_1^2}{n} + A_2 \frac{a_1^3}{n^{3/2}} + A_3 \frac{a_1^4}{n^2} + O\left(\frac{a_1^5}{n^{5/2}}\right)$$

and that

$$\psi\{w(a_1)\} = 1 + B_1 \frac{a_1}{n^{\frac{1}{2}}} + B_2 \frac{a_1^2}{n} + B_3 \frac{a_1^3}{n^{3/2}} + B_4 \frac{a_1^4}{n^2} + O\left(\frac{a_1^5}{n^{5/2}}\right).$$

So for u between $\hat{w}$ and $\hat{w}^*$,

$$\frac{1-\Phi(\hat{w}n^{\frac{1}{2}})}{1-\Phi(\hat{w}^*n^{\frac{1}{2}})} = 1 + O_p\left\{\frac{(\hat{w}-\hat{w}^*)n^{\frac{1}{2}}\phi(un^{\frac{1}{2}})}{1-\Phi(\hat{w}^*n^{\frac{1}{2}})}\right\}$$
$$= 1 + O_p\{(\hat{w}-\hat{w}^*)\hat{w}n\}$$
$$= 1 + O_p(a_1^3/n).$$

More details on the proofs of the expansions above can be obtained from the authors.

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