

Some Facts about Idempotent Matrices

1. A square matrix \mathbf{A} is *idempotent* if and only if $\mathbf{A}^2 = \mathbf{A}$.
2. If \mathbf{A} is idempotent then $\text{trace}(\mathbf{A}) = \text{rank}(\mathbf{A})$.
3. \mathbf{A} is idempotent if and only if $\text{rank}(\mathbf{A}) + \text{rank}(\mathbf{I} - \mathbf{A}) = n$ where the dimensions of \mathbf{A} are $n \times n$ and \mathbf{I} is the $n \times n$ identity matrix.
4. The matrices \mathbf{H} , $\mathbf{I} - \mathbf{H}$, $\frac{1}{n}\mathbf{J}$, and $\mathbf{H} - \frac{1}{n}\mathbf{J}$ (as defined in lecture) are idempotent. (For the last of these, it is first necessary to show that $\mathbf{HJ} = \mathbf{J}$.)
5. If $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2$ where \mathbf{A} , \mathbf{A}_1 , \mathbf{A}_2 are all idempotent then $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}_1) + \text{rank}(\mathbf{A}_2)$.