STA 257 - SOLUTIONS TO PROBLEMS ON STATISTICS HAND-OUT

#1/ $E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i E X_i = \mu \sum_{i=1}^{n} a_i$ so require $\sum_{i=1}^{n} a_i = 1$ for unbiased

#2/ $E(X_n) = \mu$ so unbiased.

For some positive number ε,

$$P(|X_n - \mu| \le \varepsilon) = P(\mu - \varepsilon < X_n < \mu + \varepsilon) = \Phi\left(\frac{\varepsilon}{\sigma}\right) - \Phi\left(-\frac{\varepsilon}{\sigma}\right)$$

does not go to 1 as $n \to \infty$ so not consistent.

$$E\left[(X_n - \bar{X})^2\right] = E(X_n^2) - 2E(X_n\bar{X}) + E(\bar{X}^2) = EX_n^2 - \frac{2}{n}\left[\sum_{i=1}^{n-1} E(X_i X_n) + EX_n^2\right] + E\bar{X}^2$$

$$= EX_n^2 - \frac{2}{n}\left[\sum_{i=1}^{n-1}(EX_i)(EX_n) + EX_n^2\right] + E\bar{X}^2$$ since X_n, X_i indept. for $i = 1, \dots, n-1$

since $X_n \sim N(\mu, \sigma^2)$, $EX_n^2 = \sigma^2 + \mu^2$

$\bar{X} \sim N(\mu, \sigma^2/n)$, $E\bar{X}^2 = \sigma^2/n + \mu^2$

$$= \sigma^2 + \mu^2 - \frac{2}{n}\left[(n-1)\mu^2 + \sigma^2 + \mu^2\right] + \frac{\sigma^2}{n} + \mu^2 = \dots = \frac{n-1}{n}\sigma^2$$

For some positive number $\varepsilon < \sigma^2$

$$P\left(|(X_n - \bar{X})^2 - \sigma^2| \le \varepsilon\right) = P\left(\sqrt{-\varepsilon + \sigma^2} \le X_n - \bar{X} \le \sqrt{\varepsilon + \sigma^2}\right)$$

$$X_n - \bar{X} = \underbrace{\left(\frac{n-1}{n}\right)X_n - \frac{1}{n}\sum_{i=1}^{n-1} X_i}_{\text{split up into linear combination of independent normal r.v.'s}} \; ; \quad \frac{n-1}{n}X_n \sim N\left(\frac{n-1}{n}\mu, \left(\frac{n-1}{n}\right)^2 \sigma^2\right)$$

$$\frac{1}{n}\sum_{i=1}^{n-1} X_i \sim N\left(\frac{n-1}{n}\mu, \frac{n-1}{n^2}\sigma^2\right)$$

so $X_n - \bar{X} \sim N\left(0, \frac{n-1}{n}\sigma^2\right)$

$$\lim_{n\to\infty} P\left(\sqrt{-\varepsilon + \sigma^2} \le X_n - \bar{X} \le \sqrt{\varepsilon + \sigma^2}\right) = \lim_{n\to\infty}\left\{\Phi\left(\frac{\sqrt{\varepsilon + \sigma^2}}{\sqrt{\frac{n-1}{n}}\,\sigma}\right) - \Phi\left(\frac{\sqrt{-\varepsilon + \sigma^2}}{\sqrt{\frac{n-1}{n}}\,\sigma}\right)\right\}$$

$$= \Phi\left(\frac{\sqrt{\varepsilon + \sigma^2}}{\sigma}\right) - \Phi\left(\frac{\sqrt{-\varepsilon + \sigma^2}}{\sigma}\right)$$ which is not 1 for $\varepsilon > 0$ so not consistent.

#3/ $EX_i = \mu$, $VX_i = \mu$ so $EX_i^2 = \mu + \mu^2$

$$E(M) = d\left[\sum_{i=1}^{n-1} EX_i^2 - \sum_{i=1}^{n-1} E(X_i X_{i+1})\right] = d\left[(n-1)(\mu + \mu^2) - (n-1)\mu^2\right]$$ since indept.

$$= d(n-1)(\mu) = \mu \text{ if } d = \frac{1}{n-1}$$

#4/ Note that this estimator doesn't make sense if n is odd.
Let $\hat{\sigma}^2$ be the estimator

$$E(\hat{\sigma}^2) = c\, E\left(\sum_{i=1}^{n} X_i^2 - 2\sum_{i=1}^{n/2} X_{2i-1}X_{2i}\right)$$

$$= c\left(n(\sigma^2+\mu^2) - 2\left(\tfrac{n}{2}\right)\mu^2\right) \quad \text{since independent}$$

$$= cn\sigma^2 = \sigma^2 \text{ if } c = \frac{1}{n}$$

#5/ Let $Y = \frac{(n-1)s^2}{\sigma^2}$, $Y \sim \chi^2_{n-1}$, $f_Y(y) = \frac{1}{2^{n-1/2}\Gamma(\frac{n-1}{2})} y^{\frac{n-1}{2}-1} e^{-y/2}$, $y \geq 0$

$= g(s)$

Find density for s: $\quad g'(s) = 2(n-1)s^2/\sigma^2$

$$f_s(s) = \frac{(n-1)^{n-1/2}}{2^{n-3/2}\sigma^{n-1}\Gamma(\frac{n-1}{2})} s^{n-2} e^{-(n-1)s^2/2\sigma^2}$$

$E(s) = \int_0^\infty s f_s(s)\,ds$ then let $u = \frac{(n-1)s^2}{2\sigma^2}$; $du = \frac{(n-1)s}{\sigma^2}ds$

$$= \cdots = \frac{\sqrt{2}\,\sigma}{\sqrt{n-1}\,\Gamma(\frac{n-1}{2})} \underbrace{\int_0^\infty u^{n/2-1} e^{-u}\,du}_{\Gamma(\frac{n}{2})}$$

#6/(a) $E(\hat{p}_2) = E\left(\frac{Y+1}{n+2}\right) = \sum_{y=0}^{n} \frac{y+1}{n+2}\binom{n}{y}p^y(1-p)^{n-y}$

$$= \frac{1}{n+2}\left\{\underbrace{\sum_{y=0}^{n} y\binom{n}{y}p^y(1-p)^{n-y}}_{=EY} + \underbrace{\sum_{y=0}^{n}\binom{n}{y}p^y(1-p)^{n-y}}_{=1}\right\}$$

$$= \frac{1}{n+2}(np+1)$$

(of course, $E\left(\frac{Y}{n+2} + \frac{1}{n+2}\right) = \frac{EY}{n+2} + \frac{1}{n+2}$!)

(b) $MSE(\hat{p}_1) = E\left(\frac{Y}{n} - p\right)^2 = \frac{1}{n^2}E(Y-np)^2 = \frac{1}{n^2}VY = \frac{1}{n^2}np(1-p)$

$MSE(\hat{p}_2) = E\left(\frac{Y+1}{n+2} - p\right)^2 = \frac{1}{(n+2)^2}(EY^2 + 2EY + 1) - 2p\,E\left(\frac{Y+1}{n+2}\right) + p^2$

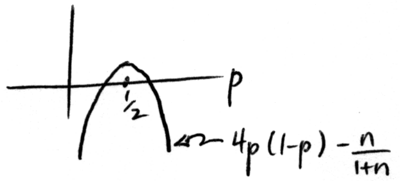
$$= \frac{1}{(n+2)^2}(np(1-p) + n^2p^2 + 2np + 1) - 2p\left(\frac{np+1}{n+2}\right) + p^2 = \cdots = \frac{np(1-p) + (1-2p)^2}{(n+2)^2}$$

#6(b) cont^d

If $MSE(\hat{p}_2) < MSE(\hat{p}_1)$, $\frac{np(1-p)+(1-2p)^2}{(n+2)^2} < \frac{p(1-p)}{n}$ $\ldots$ $\frac{n}{1+n} < 4p(1-p)$

If $4p(1-p) > 1$, $4p(1-p) > \frac{n}{1+n}$ $\left(\frac{n}{1+n} \simeq 1 \text{ for large } n\right)$

$4p(1-p) = 1$ at $p = \frac{1}{2}$, $4p(1-p) > \frac{n}{1+n}$ for values of p near $\frac{1}{2}$



#7/ Exponential $(\frac{1}{2})$ distribution is the Gamma$(1, \frac{1}{2})$ distribution (compare densities)

which is the Chi-square(2) distribution

$$\frac{X}{Y} = \frac{X/2}{Y/2} \sim F_{2,2}$$

#8/ (a) $\bar{X} \sim N(\mu, \sigma^2/n)$; $a\bar{X} + b \sim N(a\mu + b, a^2\sigma^2/n)$

(b) $\left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2_1$

(c) $Y - \mu/\sigma \sim N(0,1)$; $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$

so $$\frac{Y-\mu/\sigma}{\sqrt{(n-1)s^2/\sigma^2 \,/ n-1}} \sim t_{n-1}$$

$$= \frac{Y-\mu}{\sqrt{s^2}}$$

(d) $\left(\frac{Y-\mu}{\sigma}\right)^2 \sim \chi^2_1$; $(n-1)s^2/\sigma^2 \sim \chi^2_{n-1}$

$$\frac{(Y-\mu/\sigma)^2}{(n-1)\frac{s^2}{\sigma^2}/n-1} \sim F_{1,n-1}$$

$$= (Y-\mu)^2/s^2$$

(e) $X_i - X_j \sim N(0, 2\sigma^2)$ $\left(\frac{X_i - X_j}{\sqrt{2}\sigma}\right)^2 \sim \chi^2_1$; sum of $\frac{n}{2}$ χ^2_1 r.v.'s is a $\chi^2_{n/2}$ r.v.