STA 257 - PRACTICE PROBLEMS 9 - Solutions

#6.23 (a) For $x > \theta$

$$P(X_{(1)} \le x) = 1 - P(X_{(1)} > x) = 1 - P(\text{all of } X_1, \dots, X_n > x)$$

$$= 1 - \left[\int_x^\infty e^{-(t-\theta)}\, dt \right]^n = 1 - \left(e^{-(x-\theta)}\right)^n$$

$$f_{X_{(1)}}(x) = n e^{-n(x-\theta)} \quad \text{for } x > \theta.$$

#4/ As derived in lecture, density of $Z = X/Y$ is

$$f_Z(z) = \frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)} \frac{z^{n/2 - 1}}{(z+1)^{n+m/2}} \quad \text{for } z \ge 0$$

Density of $W = \frac{m}{n} Z$ is

$$f_W(w) = \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \left(\frac{n}{m}\right)^{n/2} \frac{w^{n/2 - 1}}{\left(\frac{n}{m} w + 1\right)^{n+m/2}}, \quad w \ge 0$$

Note that m, n are reversed in answer on problem sheet

#5/ $$EX^2 = \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \cdot x^2\, dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)} = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

$$VX = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - \left(\frac{\alpha}{\alpha+\beta}\right)^2$$

#6/ Inverse transformation: $x = \frac{1}{y+1}$

For $y > 0$, $$f_Y(y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{1}{y+1}\right)^{\alpha-1} \left(\frac{y}{y+1}\right)^{\beta-1} \left| \frac{-1}{(y+1)^2} \right|$$

#7/ Value of x that maximizes density also maximizes the ln of the density; so maximize

$$g(x) = (\alpha-1)\ln x + (\beta-1)\ln(1-x)$$

$$g'(x) = \frac{\alpha-1}{x} - \frac{\beta-1}{1-x}, \quad g''(x) < 0$$

$$g'(x) = 0 \text{ at } x = \frac{\alpha-1}{\alpha+\beta-2}.$$

#8/ As derived in lecture,

$$f_{X_{(k)}}(x) = \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n+1-k)} x^{k-1}(1-x)^{n-k}, \quad 0<x<1.$$

(a) $$EX_{(k)} = \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n+1-k)} \int_0^1 x \cdot x^{k-1}(1-x)^{n-k}\,dx$$

$$= \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n+1-k)} \frac{\Gamma(k+1)\Gamma(n-k+1)}{\Gamma(n+2)} = \frac{k}{n+1}$$

$$EX_{(k)}^2 = \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n+1-k)} \cdot \frac{\Gamma(k+2)\Gamma(n-k+1)}{\Gamma(n+3)} = \frac{k(k+1)}{(n+1)(n+2)}$$

$$VX_{(k)} = EX_{(k)}^2 - (EX_{(k)})^2 = \ldots$$

(b). Need to show $\lim_{k\to\infty} P(|X_{(k)} - \alpha| > \varepsilon) \to 0$

If $k \to \infty$, $n \to \infty$ also

By Chebyshev's inequality,

$$P(|X_{(k)} - EX_{(k)}| \geq \varepsilon) \leq \frac{k(n+1-k)}{(n+1)^2(n+2)} \Big/ \varepsilon^2$$

Take limit as $k \to \infty$, $n \to \infty$

#9/ $P(X_{(3)} < 0) = P(3 \text{ or more of } X_1, \ldots, X_{10} \text{ are } < 0)$

$$= \sum_{j=3}^{10} \binom{10}{j} [\Phi(0)]^j [1 - \Phi(0)]^{10-j}$$

where Φ is the cdf of the $N(0,1)$ distribution

$$= \left(\tfrac{1}{2}\right)^{10} \left\{ \binom{10}{3} + \binom{10}{4} + \cdots + \binom{10}{10} \right\} \doteq 0.9453$$

#10/ $$\pi(t) = \sum_{k=0}^{\infty} t^k \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} = e^{-\lambda} e^{\lambda t}$$

#12/ $$\pi(t) = \sum_{x=0}^{\infty} t^x (x+1) p^2 q^x = p^2 \sum_{x=0}^{\infty} (x+1)(tq)^x = \frac{p^2}{(1-tq)^2} \quad \text{for } |tq| < 1$$

using $$\sum_{x=0}^{\infty} (x+1) u^x = \frac{d}{du} \sum_{x=0}^{\infty} u^{x+1} = \frac{1}{(1-u)^2} \quad \text{for } |u| < 1$$

#13/ $\pi_Z(t) = \pi_X(t) \cdot \pi_Y(t)$

Z is the ~~waiting time~~
number of failures before the second success.