STA 257 – Fall 2002

Practice Problems 9

Recommended preparation for quiz to be held in tutorial on Wednesday, November 27

Material covered during the week of November 18: t distribution, F distribution, Beta distribution (4.7), Order statistics for independent uniform random variables (6.6), probability generating functions (3.10, 6.7)

Questions from the textbook (Schaeffer):

1. From Section 4.7: 4.73
2. From Chapter 4 Supplementary Exercises: 4.129
3. From Section 6.6: 6.23
4. From Section 6.7: 6.29

Additional questions:

4. Let $X \sim \chi^2(n)$ and $Y \sim \chi^2(m)$. Find the density of $W = \frac{X/n}{Y/m}$. The distribution of W is called the F distribution. [Ans: $\frac{\Gamma((m+n)/2)}{\Gamma(m/2)\Gamma(n/2)} \left(\frac{m}{n}\right)^{m/2} w^{m/2-1} \left(1 + \frac{m}{n}w\right)^{-(m+n)/2}$ for $w \geq 0$]

5. Suppose $X \sim \text{Beta}(\alpha, \beta)$. Show that the variance of X is $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

6. Let X have a $\text{Beta}(\alpha, \beta)$ distritution and let $Y = (1/X) - 1$. Find a density for Y. (The distribution of Y is called the Pareto distribution.) [Ans: $f(y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{y^{\alpha-1}}{(1+y)^{\alpha+\beta}}$ for $y > 0$]

7. Find the mode of the $\text{Beta}(\alpha, \beta)$ distribution, that is, the value of x for which the density is largest. [Ans: $(\alpha - 1)/(\alpha + \beta - 2)$]

8. Let $X_{(k)}$ the the kth order statistic of a sample of size n from the $\text{Uniform}(0, 1)$ distribution.

   (a) Find $EX_{(k)}$ and $VX_{(k)}$.

   (b) Suppose $n \to \infty$ and $k \to \infty$, but $k/(n + 1)$ is always equal to a constant α. Show that $X_{(k)} \to \alpha$ in the sense of the Weak Law of Large Numbers.

   [Ans: (a) $\frac{k}{n+1}$, $\frac{k(n+1-k)}{(n+1)^2(n+2)}$]

9. Let $X_1, X_2, \ldots, X_{10}$ be a sample from the standard normal distribution. Find the probability that the third order statistic of the sample (that is, the third smallest of the X_j's) is less than 0. [Ans: .9453]

10. Find the probability generating function of a Poisson random variable with parameter λ. [Ans: $\pi(t) = e^{\lambda t - \lambda}$]

11. $\pi(t) = \frac{1}{3}t^2(1 + 2t^3)$ is a probability generating function. What is the probability function? [Ans: There are only two possible values, 2 and 5 with probabilities 1/3 and 2/3, respectively.]

12. Let the probability function of X be $p(x) = (x+1)p^2q^x$ for $x = 0, 1, 2, \ldots$ where $q = 1-p$. (This is the distribution of the number of failures before the second success in Bernoulli trials with success probability p.) Find the probability generating function of X. [Ans: $(p/(1-qt))^2$]

13. Let X and Y be independent random variables, each having the distribution of the number of failures before the first success in Bernoulli trials – that is, the geometric distribution with parameter p, whose probability function is $p(k) = pq^k$ $(k = 0, 1, 2, \ldots)$ where $q = 1 - p$. So the probability generating function of X and that of Y are the same, $\pi(t) = p/(1 - qt)$. Find the probability generating function of the random variable $Z = X + Y$. Compare with the answer to the previous question. Why is this not surprising? [Ans: $(p/(1-qt))^2$]

Material to be covered during the week of November 25 and relevant sections in Schaeffer: more on probability generating functions (3.10, 6.7), moment generating functions (3.9, 4.10, 5.6, 6.5), Central Limit Theorem (7.4)