PRACTICE PROBLEMS 8 — STA 257 — Solutions to Additional Questions

#3/ $\dfrac{\partial(x,y)}{\partial(u,v)} = -5$

$$\iint_{D_{xy}} (x+y)\,dx\,dy = \iint_{D_{uv}} (4u+3v)\,|-5|\,du\,dv$$

$$= 5\int_0^1\int_0^1 (4u+3v)\,du\,dv = \cdots = \frac{35}{2}$$

#4/ $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3$

$$\iint_D (2x^2+xy-y^2)\,dA = \iint_D (2x-y)(x+y)\,dx\,dy$$

$$= \int_{-2}^{0}\int_1^3 uv\left(\tfrac{1}{3}\right)du\,dv = \cdots = -\frac{8}{3}$$

#5/(b) $f_X(x_1, x_2) = \lambda^2 e^{-\lambda(x_1+x_2)}$ for $x_1, x_2 > 0$

$x_1 = y_1 y_2, \quad x_2 = y_2 - y_1 y_2$

$J(y_1, y_2) = \begin{vmatrix} y_2 & y_1 \\ -y_2 & (1-y_1) \end{vmatrix} = y_2$

$f_Y(y_1, y_2) = \lambda^2 y_2 e^{-\lambda y_2}$ for $0 < y_1 < 1,\ y_2 > 0$ (0 otherwise)

#6/(a) $f_X(x_1, x_2) = \dfrac{1}{2\pi} e^{-(x_1^2+x_2^2)/2}$ for $x_1, x_2 \in \mathbb{R}$.

$x_1 = \dfrac{y_1+y_2}{2}, \quad x_2 = \dfrac{y_1-y_2}{2}$

$J(y_1, y_2) = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}$

For $y_1, y_2 \in \mathbb{R}$ $\quad f_Y(y_1, y_2) = \dfrac{1}{2\pi} e^{-\frac14(y_1^2+2y_1y_2+y_2^2+y_1^2-2y_1y_2+y_2^2)/2}\left|\dfrac{1}{2}\right|$

$= \cdots$

#7/ Note: question should read that the means of X_1 and X_2 are $1/\lambda$, $1/\mu$ respectively.

$$f_X(x_1, x_2) = \lambda\mu e^{-\lambda x_1 - \mu x_2}, \quad x_1, x_2 > 0$$

Inverse transformation: $x_2 = y_2/\mu$, $x_1 = (y_1 - y_2)/\lambda$

$$J(y_1, y_2) = \begin{vmatrix} 1/\lambda & -1/\lambda \\ 0 & 1/\mu \end{vmatrix} = \frac{1}{\lambda\mu}$$

For $0 < y_2 < y_1$

$$f_Y(y_1, y_2) = \lambda\mu e^{-y_1} \left|\frac{1}{\lambda\mu}\right| = e^{-y_1}$$

$$f_{Y_1}(y_1) = \int_0^{y_1} e^{-y_1}\, dy_2 = y_1 e^{-y_1} \text{ for } y_1 > 0 \text{ (0 otherwise).}$$

Other ways you could solve this problem:

$\lambda X_1 \sim$ Exponential (1) and $\mu X_2 \sim$ Exponential (1) (show this)

If $Y = \lambda X_1 + \mu X_2$, $P(Y \le y) = \int_0^y \int_0^{y - x_2} e^{-x_1 - x_2}\, dx_1\, dx_2$ etc.

and differentiate to get result.

or use the fact that an exponential (1) r.v. has a Gamma (1,1) distribution and the sum of two Gamma (1,1) r.v.'s has a Gamma (2,1) distribution

#8/ (a) Z_1, Z_2 independent

(b) $J(z_1, z_2) = \sigma_1\sigma_2$

$$f_X(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)\right] \Big/ 2(1-\rho^2)\right\}$$

for $x_1, x_2 \in \mathbb{R}$.

#10/ $f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x \geq 0$

This is maximized at the value of x that maximizes $g(x) = \log\left(x^{\alpha-1} e^{-\lambda x}\right)$

$$= (\alpha-1)\log x - \lambda x$$

$g'(x) = \frac{\alpha-1}{x} - \lambda$ which is 0 at $x = \frac{\alpha-1}{\lambda}$

$g''(x) = -\frac{\alpha-1}{x^2} < 0$ since $\alpha \geq 1$

So the mode is at $x = \frac{\alpha-1}{\lambda} < \frac{\alpha}{\lambda} = EX$.