STA 257 – Fall 2002

Practice Problems 8

Recommended preparation for quiz to be held in tutorial on Wednesday, November 20

Sections from Schaeffer covered the week of November 11: Gamma distribution (4.5) and Chi-square distribution, Multivariate change of variables theorem, t distribution, F distribution, Beta distribution (4.7)

Questions from the textbook (Schaeffer):

1. From Section 4.5: 4.47, 4.53
   Before doing these questions, you should derive the expected value and variance of a random variable which follows a Gamma distribution.

2. From Section 4.7: 4.73

Additional questions:

3. Evaluate
   $$\int\int_{D_{xy}} (x+y)dxdy$$
   where D_{xy} is the parallelogram with vertices $(0,0)$, $(2,1)$, $(1,3)$, and $(3,4)$. *Hint*: The transformation $x = u + 2v$, $y = 3u + v$ maps the unit square in the uv-plane onto D_{xy}. [Ans: 35/2]

4. Evaluate
   $$\int\int_D (2x^2 + xy - y^2)dA$$
   where D is the set of points (x, y) such that $1 \le 2x - y \le 3$ and $-2 \le x + y \le 0$. Note that this integral could be done without changing variables, but to do so it is necessary to decompose D into subregions (why?) and to do some messy polynomial integration. For a simpler approach, use the transformation $u = 2x - y$ and $v = x + y$. [Ans: $-8/3$]

5. Let X_1 and X_2 be independent, each having the exponential distribution with mean $1/\lambda$. Define $Y_1 = X_1/(X_1 + X_2)$ and $Y_2 = X_1 + X_2$. The main point of this exercise is to find the distribution of Y_1, but we can learn other things as well.
   (a) Write down the set of possible values of the pair (Y_1, Y_2).
   (b) Find a joint density for Y_1 and Y_2.
   (c) Find the marginal densities for Y_1 and Y_2. What is the name of the distribution of Y_1?
   (d) Are Y_1 and Y_2 independent?

   [Ans: (a) $0 < y_1 < 1$, $y_2 > 0$ (b) $f_Y(y_1, y_2) = \lambda^2 y_2 e^{-\lambda y_2}$ for y_1 and y_2 as given in (a) (c) $f_{Y_1}(y_1) = 1$ for $0 < y_1 < 1$ (uniform on $(0, 1)$); $f_{Y_2}(y_2) = \lambda^2 y_2 e^{-\lambda y_2}$ for $y_2 > 0$ (d) Yes]

6. Let X_1 and X_2 be independent, each having the standard normal distribution. Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$.

   (a) Find a joint density for Y_1 and Y_2.

   (b) Find the marginal densities of Y_1 and Y_2. Identify the distributions by name.

   (c) Are Y_1 and Y_2 independent?

   [Ans: (a) $f_Y(y_1, y_2) = \frac{1}{4\pi} e^{-(y_1^2+y_2^2)/4}$ for y_1, y_2 any real numbers (b) Each is $\frac{1}{\sqrt{4\pi}} e^{-y^2/4}$ for y any real number, i.e. $N(0,2)$ (c) Yes]

7. Let X_1 and X_2 be independent, each with an exponential distribution. Let X_1 have mean λ and X_2 have mean μ. Find a density for $\lambda X_1 + \mu X_2$. (*Hint:* Let this equal Y_1 and let $Y_2 = \mu X_2$.) [Ans: $f_{Y_1} = y_1 e^{-y_1}$ for $y_1 > 0$]

8. Suppose Z_1 and Z_2 are both standard normal random variables but they are not independent. Let ρ be the correlation coefficient of Z_1 and Z_2. The joint density of Z_1 and Z_2 is

$$f(z_1, z_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-(z_1^2+z_2^2-2\rho z_1 z_2)/2(1-\rho^2)}$$

   (a) What is true if $\rho = 0$?

   (b) Find the general bivariate normal density for random variables $X_1 = \sigma_1 Z_1 + \mu_1$ and $X_2 = \sigma_2 Z_2 + \mu_2$.

9. (a) The $\chi^2(2)$ distribution is also known by another name. What is it?

   (b) Let the coordinates X and Y of a random point in the plan be independent $N(0,1)$ random variables, and let W be the squared distance of the random point from the origin. Without doing any calculations, name the distribution of W.

   [Ans: (a) exponential with mean 2 (b) exponential with mean 2]

10. Let $f(x)$ be the Gamma(α, λ) density. Suppose $\alpha > 1$. For which value of x is this density largest? This number is called the mode of the distribution. Is the mode larger or smaller than the expected value of the distribution? [Ans: $(\alpha - 1)/\lambda$ (smaller than the expected value)]

Material to be covered during the week of November 18 and relevant sections in Schaeffer: probability generating functions (3.10, 6.7), moment generating functions (3.9, 4.10, 5.6, 6.5), Central Limit Theorem (7.4)