PRACTICE PROBLEMS 7

Solutions to Additional Questions

#4/ $EY = E(E(Y|X)) = E(\mu X) = \mu EX = \mu\left(\frac{1}{\lambda}\right)$ (Correction to given answer)

$VY = V(E(Y|X)) + E(V(Y|X)) = V(\mu X) + E(\sigma^2 X^2)$

$= \mu^2 VX + \sigma^2 EX^2 = \mu^2\left(\frac{1}{\lambda^2}\right) + \sigma^2\left(\frac{2}{\lambda^2}\right)$

#5/ For $y>0$, $f_Y(y) = f_X(\ln y)\left|\frac{d}{dy}\ln y\right| = \frac{1}{\sqrt{2\pi}} e^{-(\ln y)^2/2}\,\frac{1}{y}$

For $y \le 0$, $f_Y(y) = 0$

#6/ If $y<0$, $F_Y(y) = 0$

If $0 \le y < 1$, $F_Y(y) = P(X^2/4 < y) = P(X < 2\sqrt{y}) = y$

If $y \ge 1$, $F_Y(y) = 1$

#7/(a) $X = F_Y(y)$

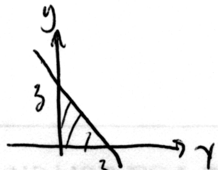
For $y \in \mathbb{R}$, $f_Y(y) = f_X(F_Y(y))\left|\frac{d}{dy}F_Y(y)\right|$ Apply change-of-variable theorem

$= 1 \cdot f_Y(y)$

(b) $F_Y(y) = \int_1^y \frac{2}{t^3}\,dt = 1 - \frac{1}{y^2}$ for $y>1$ (0 for $y \le 1$)

For $x \in [0,1]$, inverse of this function is $y = \frac{1}{\sqrt{1-x}}$

#8/ (a)



For $z>0$,

$F_Z(z) = P(X+Y \le z) = \int_0^z \int_0^{z-y} 6e^{-3x}e^{-2y}\,dx\,dy$

$= \cdots = 2e^{-3z} - 3e^{-2z}$

differentiate for density

(b) For $x>0$, $f_X(x) = 3e^{-3x}$; For $y>0$, $f_Y(y) = 2e^{-2y}$

By convolution theorem,

for $z>0$, $f_Z(z) = \int_0^z 3e^{-3x}\cdot 2e^{-2(z-x)}\,dx = \cdots = 6e^{-2z} - 6e^{-3z}$

#9/ (b) $-2Y \sim N(24, 60)$ so $X - 2Y \sim N(39, 90)$

(c) $2X \sim N(30, 120)$, $3Y \sim N(-36, 135)$ so $2X + 3Y \sim N(-6, 255)$

$$P(2X + 3Y \leq 5) = P\left(Z \leq \frac{5+6}{\sqrt{255}}\right) \text{ where } Z \sim N(0,1)$$

(d) $X - Y \sim N(27, 45)$; $P(X > Y) = P(X - Y > 0)$

$$= P\left(Z > \frac{0-27}{\sqrt{45}}\right) \text{ where } Z \sim N(0,1)$$

$$\doteq 0.99997$$

#10/ (a) $\bar{X}_{30} \sim N(20, 25/30)$

$$P(19 < \bar{X}_{30} < 21) = P\left(Z < \frac{21-20}{\sqrt{5/6}}\right) - P\left(Z \leq \frac{19-20}{\sqrt{5/6}}\right)$$

where $Z \sim N(0,1)$.

(b) $S_{30} \sim N(600, 750)$; $P(S_{30} > 650) = P\left(Z > \frac{650-600}{\sqrt{750}}\right)$

#11/ Suppose $X \sim N(\mu, \sigma^2)$.
Want to find a such that

$$0.95 = P(\mu - a\sigma \leq X \leq \mu + a\sigma)$$

$$= P(Z \leq a) - P(Z < -a) \text{ where } Z \sim N(0,1)$$

$$= 2\left[P(Z \leq a) - \tfrac{1}{2}\right]$$

From Table 4 in Schaeffer

$$P(Z \leq a) - \tfrac{1}{2} = 0.475 \text{ for } a = 1.96$$