PRACTICE PROBLEMS 5
Solutions to Additional Questions

#5.8 from Schaeffer

(a) $f_1(x_1) = \begin{cases} \int_0^{1-x_1} 2\,dx_2 = 2(1-x_1) & 0 \le x_1 \le 1 \\ 0 & \text{otherwise.} \end{cases}$

$f_2(x_2) = \begin{cases} \int_0^{1-x_2} 2\,dx_1 = 2(1-x_2) & 0 \le x_2 \le 1 \\ 0 & \text{otherwise} \end{cases}$

(b) Note that $f_1(x_1)f_2(x_2) = 2(1-x_1)2(1-x_2) \neq 2 = f(x_1,x_2)$ for $0 \le x_1 \le 1-x_2$ so X_1, X_2 are not independent

(c) $P(X_1 > \frac{1}{2} \mid X_2 = \frac{1}{4}) = \int_{1/2}^{1} f(x_1 \mid x_2 = \frac{1}{4})\,dx_1$

$f(x_1 \mid x_2) = \frac{2}{2(1-x_2)}$

$= \int_{1/2}^{1} \frac{1}{1-1/4}\,dx_1 = \frac{2}{3}$

#10 from Practice Problems 4

P(family has 2 girls)

$= P(2 \text{ girls and } 2 \text{ children}) + P(2 \text{ girls and } 3 \text{ children}) + \cdots$

$= P(2 \text{ girls} \mid 2 \text{ children})P(2 \text{ children}) + P(2 \text{ girls} \mid 3 \text{ children}) P(3 \text{ children})$

$+ P(2 \text{ girls} \mid 4 \text{ children})P(4 \text{ children}) + \cdots$

$$= \left(\tfrac{1}{2}\right)^2 \frac{(1.8)^2 e^{-1.8}}{2!} + 3\left(\tfrac{1}{2}\right)^3 \frac{(1.8)^3 e^{-1.8}}{3!} + \cdots + \binom{k}{2}\left(\tfrac{1}{2}\right)^k \frac{(1.8)^k e^{-1.8}}{k!} + \cdots$$

$$= e^{-1.8}\left(\tfrac{1}{2}\right)^2 \frac{(1.8)^2}{2!} \underbrace{\left\{\sum_{k=2}^{\infty} \frac{1}{(k-2)!}(.9)^{k-2}\right\}}_{e^{.9}} = \frac{(.9)^2 e^{-.9}}{2!}$$

#11 from Practice Problems 4

(a) $P(X=2 \text{ and } Y=2) = P(X=2)P(Y=2) = (.5)^2$

(b) $P(X=Y) = P(X=Y=0) + P(X=Y=1) = P(X=Y=2) = (.2)^2 + (.3)^2 + (.5)^2$

(c) $P(XY=0) = P(X=0 \text{ or } Y=0) = P(X=0) + P(Y=0) - P(X=Y=0)$

$= .2 + .2 - (.2)^2$

#12 from Practice Problems 4

(a) If A is favourable for B, then $P(B|A) > P(B)$ so $\frac{P(AB)}{P(A)} > P(B)$

$P(AB) > P(A)P(B)$

If $P(AB) > P(A)P(B)$, then $\frac{P(AB)}{P(A)} > P(B)$, so $P(B|A) > P(B)$

so A is favourable for B

(b) If A is favourable for B, from (a) $P(AB) > P(A)P(B)$

so $\frac{P(AB)}{P(B)} > P(A)$ so $P(A|B) > P(A)$ so B is favourable for A

(c) If A is unfavourable for B, $P(B|A) < P(B)$

then $1 - P(B|A) > 1 - P(B)$

$P(\bar{B}|A) > P(\bar{B})$ so A is favourable for $\bar{B}$

#5 from Practice Problems 5

Y \ X	0	1	2	3	Marginal for Y
0	1/16	1/16	0	0	1/8
1	1/16	3/16	2/16	0	3/8
2	0	2/16	3/16	1/16	3/8
3	0	0	1/16	1/16	1/8
Marginal for X	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	

(d) Not independent since, for example, $P(X=2, Y=0)$

$\neq P(X=2)P(Y=0)$

(e) $P(X=0|Y=2) = \frac{P(X=0, Y=2)}{P(Y=2)} = 0$

$P(X=1|Y=2) = \frac{P(X=1, Y=2)}{P(Y=2)}$

$= \frac{2/16}{3/8} = \frac{1}{3}$ etc.

#6 from Practice Problems 5

(a) $p_X(x) = \sum_{y=x}^{\infty} \frac{(\lambda/2)^y e^{-\lambda}}{x!\,(y-x)!} = \frac{(\frac{\lambda}{2})^x e^{-\lambda}}{x!} \cdot \sum_{y=x}^{\infty} \frac{(\frac{\lambda}{2})^{y-x}}{(y-x)!}$

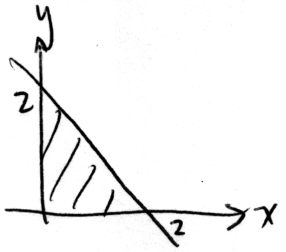
$= \frac{(\lambda/2)^x e^{-\lambda}}{x!} e^{\lambda/2}$ Poisson density with parameter $\lambda/2$

(b) $p_Y(y) = \sum_{x=0}^{y} \frac{(\lambda/2)^y e^{-\lambda}}{x!\,(y-x)!} = \frac{\lambda^y e^{-\lambda}}{y!} \underbrace{\sum_{x=0}^{y} \binom{y}{x} \left(\frac{1}{2}\right)^y}_{=1}$ Poisson density with parameter λ

(c) No $p_X(x)\, p_Y(y) \neq p(x,y)$

#7 from Practice Problems 5

(a)



(b) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dy\, dx = \int_0^2 \int_0^{2-x} c\, x^2 y\, dy\, dx$

$= c \int_0^2 x^2 \frac{(2-x)^2}{2} dx = c \frac{8}{15}$ for integral to be 1, c must be $\frac{15}{8}$

(c) No. Region where $f(x,y)$ is positive cannot be expressed as Cartesian product of a set of points for X and for Y

(d) $f_X(x) = \int_0^{2-x} \frac{15}{8} x^2 y\, dy = \frac{15}{16} x^2 (2-x)^2$

(f) $f_Y(y) = \int_0^{2-y} \frac{15}{8} x^2 y\, dx = \frac{5}{8} y (2-y)^3$

#8 from Practice Problems 5

Use the following two facts:

$P(X<Y) + P(X=Y) + P(X>Y) = 1$

and because X, Y are independent with the same distribution,

$P(X<Y) = P(Y<X)$.

#9 from Practice Problems 5

(a) $f_X(x) = \int_0^\infty \lambda^3 x e^{-\lambda(x+y)}\, dy = \dots = \lambda^2 x e^{-\lambda x}$ for $x > 0$ and 0 otherwise.

$f_Y(y) = \int_0^\infty \lambda^3 x e^{-\lambda(x+y)}\, dx$ (integrate by parts)

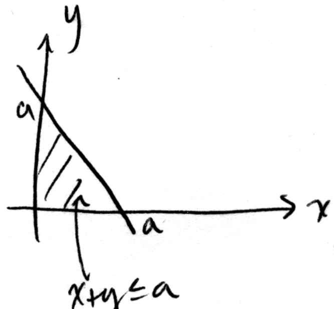
$= \lambda e^{-\lambda y}$, $y > 0$ and 0 otherwise.

X, Y independent since $f_X(x) f_Y(y) = f(x, y)$

(b) To get answer, evaluate $\int_0^b \int_0^a \lambda^3 x e^{-\lambda(x+y)}\, dx\, dy$

(c) Evaluate $\int_0^a \lambda^2 x e^{-\lambda x}\, dx$

(d)



Evaluate $\int_0^a \int_0^{y-a} \lambda^3 x e^{-\lambda(x+y)}\, dx\, dy$

(e) From (d), Z has distribution function

$F_Z(z) = 1 - \left(1 + \lambda z + \frac{\lambda^2 z^2}{2}\right) e^{-\lambda z}$

get the density by differentiating