STA 257 – Solutions to additional questions from Practice Problems 10

5.68 (a) $M_{X_1,X_2,X_3}(t,t,t) = E\left(e^{t(X_1+X_2+X_3)}\right) = M_{X_1+X_2+X_3}(t)$

(b) $M_{X_1,X_2,X_3}(t,t,0) = E\left(e^{t(X_1+X_2)}\right) = M_{X_1+X_2}(t)$

(c) $$\left.\frac{\partial^{k_1+k_2+k_3} M(t_1,t_2,t_3)}{\partial t_1^{k_1}\partial t_2^{k_2}\partial t_3^{k_3}}\right|_{t_1=t_2=t_3=0} = \left.\frac{E\left(\partial^{k_1+k_2+k_3} e^{t_1X_1+t_2X_2+t_3X_3}\right)}{\partial t_1^{k_1}\partial t_2^{k_2}\partial t_3^{k_3}}\right|_{t_1=t_2=t_3=0}$$

$$= E\left(X_1^{k_1}X_2^{k_2}X_3^{k_3}e^{t_1X_1+t_2X_2+t_3X_3}\right)\Big|_{t_1=t_2=t_3=0}$$

$$= E\left(X_1^{k_1}X_2^{k_2}X_3^{k_3}\right)$$

7.40 Let $X_i \sim \chi^2_{(1)}$ and $X_1,\ldots,X_n$ be independent.

$Y = \sum_{i=1}^{n} X_i \sim \chi^2(n)$

You can use mgfs to show this:

$$m_Y(t) = \int_0^\infty e^{yt}\frac{1}{2^{n/2}\Gamma(\frac{n}{2})}y^{n/2-1}e^{-y/2}\,dy = \cdots = \left[\frac{1}{2(\frac{1}{2}-t)}\right]^{n/2}$$
$$= [m_{X_i}(t)]^n$$
$$= m_{\Sigma X_i}(t).$$

$\bar{X} = Y/n$

By the Central Limit Theorem,

$\dfrac{\sqrt{n}\,(\bar{X}-E(X_i))}{\sqrt{V(X_i)}}$ converges in distribution to a standard normal r.v.

$$= \frac{n}{n} * (\quad '' \quad) = \frac{n}{n}\left(\frac{\sqrt{n}\,(\bar{X}-1)}{\sqrt{2}}\right)$$ since can show $EX_i = 1$, $VX_i = 2$

$$= \frac{Y-n}{\sqrt{2n}}$$

#5/ $\pi'(t) = \dfrac{2qp^2}{(1-qt)^3}$, $\pi''(t) = \dfrac{6q^2p^2}{(1-qt)^4}$

$\pi'(1) = \dfrac{2q}{p}$, $\pi''(1) = \dfrac{6q^2}{p^2}$

$= EX$

$VX = \dfrac{6q^2}{p^2} + \dfrac{2q}{p} - \left(\dfrac{2q}{p}\right)^2$

#6/ Maclaurin series of $\pi(t)$ is $\pi(t) = 2C\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \dots\right)$

(a) $P(X=1) = 0$, $P(X=10) = \dfrac{2C}{10!}$

(b) $\pi'(t) = C(e^t - e^{-t})$; $\pi'(1) = C(e^1 - e^{-1})$

(c) $\pi(1) = 1 \Rightarrow 1 = 2C(e + e^{-1})$

#7/ (a) $m_Z(t) = e^{t^2/2}$; $m_X(t) = e^{\mu t} e^{\sigma^2 t^2/2}$

$m_X'(t) = \mu e^{\mu t} + \sigma^2 t e^{\sigma^2 t^2/2}$; $m_X'(0) = \mu = EX$

$m_X''(t) = \mu^2 e^{\mu t} + \sigma^2 e^{\sigma^2 t^2/2} + \sigma^4 t^2 e^{\sigma^2 t^2/2}$; $m_X''(0) = \mu^2 + \sigma^2$

$VX = \mu^2 + \sigma^2 - (\mu^2)$

#8/ (a) $m(t) = \sum_{k=0}^{\infty} \dfrac{e^{-\lambda}(e^t\lambda)^k}{k!} = e^{-\lambda} e^{\lambda e^t}$

(b) $\pi(t) = e^{\lambda t - \lambda}$

$m(t) = \pi(e^t) = e^{\lambda e^t - \lambda}$

#9. Let $\bar{X}$ be the average of the 75 deposits

By the CLT, $\bar{X} \approx N\left(638, \frac{(126)^2}{75}\right)$

$P(\bar{X} > 671) \approx P\left(Z > \frac{671-638}{126/\sqrt{75}}\right)$ where $Z \sim N(0,1)$

$= 0.0116$

#10. (b) Chebyshev's inequality:

$P(|\bar{X}_{50} - E\bar{X}_{50}| \geq \varepsilon) \leq \frac{V\bar{X}_{50}}{\varepsilon^2}$

$P(\bar{X}_{50}$ fails to be between 98 and 102$)$

$= P(|\bar{X}_{50} - 100| \geq 2)$

$\leq \frac{1/2}{2^2}$

(c) By the CLT, $\bar{X}_{50} \approx N(100, \frac{1}{2})$

$P(\bar{X}_{50}$ fails to be between 98 and 102$)$

$= 1 - P(98 \leq \bar{X}_{50} \leq 102)$

$\approx 1 - \left[\Phi\left(\frac{102-100}{\sqrt{1/2}}\right) - \Phi\left(\frac{98-100}{\sqrt{1/2}}\right)\right]$

$= .0046$

where Φ is the cdf of the standard normal distribution