STA 257~~8~~F - Fall, 2002
Term Test
October 28, 2002

INSTRUCTIONS:

- Time: 105 minutes
- No aids allowed.
- Answers that are algebraic expressions should be simplified. Series and integrals should be evaluated. Numerical answers need not be expressed in decimal form.
- Total points: 70

NAME: SOLUTIONS

STUDENT NUMBER:

TUTORIAL: (circle one)

A	B	C	D	E
LM 123	UC 256	WI 523	UC 152	UC 328
Mohammad D.	Mohammed S.	Shuying	Yan	Xiaobin

1	2	3	4	5	6	7	8	9

1. (10 points)

(a) $(S, \mathcal{F}, \mathcal{P})$ is a probability space. Assume $\mathcal{F}$ is a valid event space. State the three conditions P must satisfy to be a valid probability measure.

(1) $P(S) = 1$

(2) For any $A \in \mathcal{F}$, $P(A) \geq 0$

(3) For $A_1, A_2, A_3, \ldots \in \mathcal{F}$ and disjoint

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

(b) Show how the conditions in (a) result in $P(\overline{A}) = 1 - P(A)$ where A is a set in $\mathcal{F}$ and $\overline{A}$ is the complement of A.

$A \cup \bar{A} = S$

and $A, \bar{A}$ are disjoint

so $P(A \cup \bar{A}) = P(A) + P(\bar{A}) = 1$

2. (5 points) In Bernoulli trials with success probability $\frac{1}{4}$, the probability the first success comes on the kth trial is $\frac{1}{4}\left(\frac{3}{4}\right)^{k-1}$. Find the probability that the first success comes on a trial whose number is divisible by 3.

P(1^{st} success on 3^{rd} or 6^{th} or 9^{th} or ... trial)

$$= \frac{1}{4}\left(\frac{3}{4}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^5 + \frac{1}{4}\left(\frac{3}{4}\right)^8 + \cdots$$

$$= \frac{1}{4}\left(\frac{3}{4}\right)^2 \left[\frac{1}{1-(3/4)^3}\right]$$

$$= \frac{9}{64}\left[\frac{1}{1-27/64}\right]$$

$$= \frac{9}{37}$$

3. (5 points) Prove the following: if events A and B are independent, then their complements, $\overline{A}$ and $\overline{B}$, are independent.

$P(A) = P(A\bar{B}) + P(AB)$

$= P(A\bar{B}) + P(A)P(B)$ since A, B independent

so $P(A\bar{B}) = P(A)(1-P(B))$

$= P(A)\,P(\bar{B})$ so $A, \bar{B}$ independent

$P(\bar{B}) = P(A\bar{B}) + P(\bar{A}\bar{B})$

$= P(A)\,P(\bar{B}) + P(\bar{A}\bar{B})$ from above

so $P(\bar{A}\bar{B}) = P(\bar{B})(1-P(A))$

$= P(\bar{B})\,P(\bar{A})$

So $\bar{A}, \bar{B}$ are independent

4. (7 points) The entire output of a factory is produced on three machines which account for 20%, 30%, and 50% of the output, respectively. The fraction of defective items produced is 5% for the first machine, 3% for the second, and 1% for the third.

(a) What is the probability that a randomly chosen item produced in this factory is defective?

Let M_i be the event the item was produced on the i^{th} machine.

By the Law of Total Probability:

$$P(\text{defective}) = P(\text{defective} \mid M_1)\,P(M_1) + P(\text{defective} \mid M_2)\,P(M_2) + P(\text{defective} \mid M_3) \cdot P(M_3)$$

$$= .05(.2) + .03(.3) + .01(.5)$$

$$= .024$$

(b) If an item is chosen at random from the total output and is found to be defective, what is the probability that it was made by the third machine?

$$P(M_3 \mid \text{defective}) = \frac{P(\text{defective} \mid M_3)\,P(M_3)}{P(\text{defective})}$$

$$= \frac{.01(.5)}{.024}$$

$$= .2083$$

5. (8 points) Suppose a continuous random variable X has density function $f(x) = xe^{-x^2/2}$ for $x > 0$ and 0 otherwise.

(a) Verify that $f(x)$ is a valid density function.

- $f(x) \geq 0$ for all $x \in \mathbb{R}$
- $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} x e^{-x^2/2}\,dx$

$$= -e^{-x^2/2}\Big|_0^{\infty}$$

$$= 1$$

(b) Find the distribution function for X.

For $x > 0$,

$$F(x) = P(X \leq x) = \int_0^x t e^{-t^2/2}\,dt$$

$$= -e^{-t^2/2}\Big|_0^x$$

$$= 1 - e^{-x^2/2}$$

$$\text{so } F(x) = \begin{cases} 1 - e^{-x^2/2} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

6. (5 points) X is a continuous random variable with possible values $\{x : 0 < x < \alpha\}$, $\alpha < \infty$ and density function $f(x)$ and distribution function $F(x)$. Prove

$$EX = \int_0^\alpha (1 - F(t))\, dt$$

(*Hint*: One possible method is to use integration by parts.)

ONE POSSIBLE SOLUTION:

$$EX = \int_0^\alpha x f(x)\, dx$$

Let $x = u$ $\quad dv = f(x)dx$

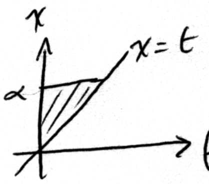
$dx = du$ $\quad v = F(x)$

$$= xF(x)\Big|_0^\alpha - \int_0^\alpha F(x)\, dx$$

$$= \alpha - \int_0^\alpha F(x)\, dx$$

$$= \int_0^\alpha (1 - F(t))\, dt$$

ANOTHER POSSIBLE SOLUTION:



$$EX = \int_0^\alpha x f(x)\, dx$$

$$= \int_0^\alpha \int_0^x dt\, f(x)\, dx$$

$$= \int_0^\alpha \int_t^\alpha f(x)\, dx\, dt$$

$$= \int_0^\alpha (F(\alpha) - F(t))\, dt$$

$$= \int_0^\alpha (1 - F(t))\, dt$$

7. (5 points) Suppose X is a random variable with finite expectation and a is a real number. Show that if $P(X \le a) = 1$ then $EX \le a$. You may asssume that any properties of expectation given in class are known.

ONE POSSIBLE SOLUTION:

If $P(X \le a) = 1$

then $a - X$ is a non-negative r.v.

So $E(a - X) \ge 0$

So $a \ge EX$

ANOTHER POSSIBLE SOLUTION:

Since $P(X \le a) = 1$

$$EX = \int_{-\infty}^a t f(t)\, dt$$

if X is a continuous r.v. with density f

$$\le \int_{-\infty}^a a f(t)\, dt$$

$$= a \underbrace{\int_{-\infty}^a f(t)\, dt}_{\le 1 \text{ since } \int_{-\infty}^\infty f(t)\,dt = 1}$$

$$\le a$$

And show a similar result for X discrete with probability function p

8. (10 points) Let X and Y be discrete random variables with joint probability function given by

$$p_{X,Y}(3,5) = 3/8; \quad p_{X,Y}(3,11) = 1/4;$$
$$p_{X,Y}(6,5) = 1/8; \quad p_{X,Y}(6,11) = 1/8;$$
$$p_{X,Y}(8,5) = 1/8$$

with $p_{X,Y}(x,y) = 0$ for other values of (x,y).

(a) Compute the marginal probability functions $p_X(x)$ and $p_Y(y)$.

$p_X(3) = \frac{3}{8} + \frac{1}{4} = \frac{5}{8}$

$p_X(6) = \frac{1}{8} + \frac{1}{8} = \frac{2}{8}$

$p_X(8) = \frac{1}{8}$; $p_X(x) = 0$ for $x \neq 3,6,8$

$p_Y(5) = \frac{3}{8} + \frac{1}{8} + \frac{1}{8} = \frac{5}{8}$

$p_Y(11) = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$

$p_Y(y) = 0$ for $y \neq 5,11$

(b) Compute the expected value EX.

$$EX = 3\left(\frac{5}{8}\right) + 6\left(\frac{2}{8}\right) + 8\left(\frac{1}{8}\right)$$

$$= \frac{35}{8}$$

(c) Compute $P(X+Y > 12)$.

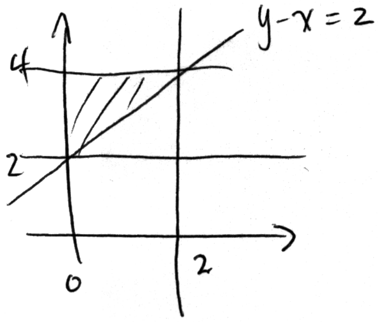
$$P(X+Y=12) = p_{X,Y}(3,11) + p_{X,Y}(6,11) + p_{X,Y}(8,5)$$

$$= \frac{1}{4} + \frac{1}{8} + \frac{1}{8}$$

$$= \frac{1}{2}$$

9. (15 points) Suppose continuous random variables X, Y have the joint density function $f(x,y) = \frac{1}{8}(6 - x - y)$, $0 < x < 2$, $2 < y < 4$ and 0 elsewhere.

(a) Find $P(Y - X \overset{>}{\cancel{<}} 2)$.



$$P(Y-X>2) = \int_2^4 \int_0^{y-2} \frac{1}{8}(6-x-y)\,dx\,dy$$

$$= \frac{1}{8}\int_2^4 \left(6x - \frac{x^2}{2} - xy\right)\Big|_{x=0}^{y-2} dy$$

$$= \frac{1}{8}\int_2^4 \left(10y - 14 - \frac{3}{2}y^2\right) dy$$

$$= \frac{1}{8}\left(5y^2 - 14y - \frac{y^3}{2}\right)\Big|_{y=2}^{4}$$

$$= \cdots = \frac{1}{2}$$

(b) Find the marginal density function for Y.

$$f_Y(y) = \int_0^2 \frac{1}{8}(6-x-y)\,dx$$

$$= \frac{1}{8}\left(6x - \frac{x^2}{2} - xy\right)\Big|_{x=0}^{2}$$

$$= \frac{1}{8}(12 - 2 - 2y)$$

$$= \frac{1}{4}(5-y) \text{ for } 2<y<4 \text{ and } 0 \text{ otherwise.}$$

(c) Find the conditional probability that X is less than 1 given that $Y = 3$.

$$P(X<1 \mid Y=3) = \int_0^1 \frac{\frac{1}{8}(6-x-3)}{\frac{1}{4}(5-3)}\,dx$$

$$= \int_0^1 \frac{1}{4}(3-x)\,dx$$

$$= \frac{1}{4}\left(3x - x^2/2\right)\Big|_{x=0}^{1} = \frac{1}{4}\left(\frac{5}{2}\right) = \frac{5}{8}$$

(d) Are X and Y independent? Why or why not?

No, density cannot be factored.