F02

STA 257S - Summer, 1996
Test #2
July 29, 1996

INSTRUCTIONS:

- Time: 50 minutes
- No aids allowed.
- Answers that are algebraic expressions should be simplified. Series and integrals should be evaluated. Numerical answers need not be expressed in decimal form.
- Total points: 35

NAME: SOLUTIONS

STUDENT NUMBER:

TUTOR:

1. (5 points) A student takes a multiple choice test. Each question has four possible answers. She knows the answers to 50% of the questions, can narrow the choice down to two answers 30% of the time, and does not know anything about the remaining 20% of the questions. What is the probability that she'll correctly answer a question chose at random from the test?

(Law of Total Probability)

$$P(\text{correct}) = P(\text{correct} \mid \text{knows}) \cdot P(\text{knows}) + P(\text{correct} \mid \text{narrowed}) \cdot P(\text{narrowed}) + P(\text{correct} \mid \text{guesses}) \cdot P(\text{guesses})$$

$$= 1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{3}{10} + \frac{1}{4} \cdot \frac{1}{5}$$

$$= \frac{7}{10}$$

2. (5 points) A, B, and C are events with $P(A) = 0.3$, $P(B) = 0.4$, $P(C) = 0.5$, A and B are disjoint, A and C are independent, and $P(B|C) = 0.1$. Find $P(A \cup B \cup C)$.

$P(AB) = 0$, $P(ABC) = 0$

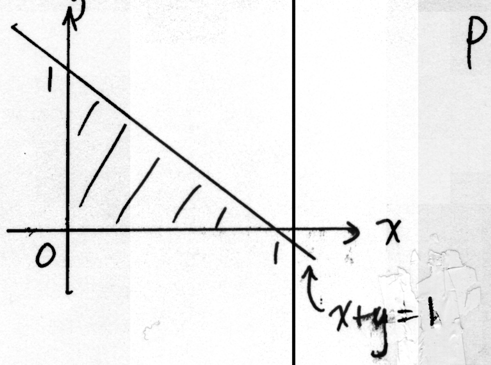
$P(AC) = P(A)P(C) = 0.15$, $P(BC) = P(B/C)P(C) = 0.05$

$$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$$
$$= 0.3 + 0.4 + 0.5 - 0 - 0.15 - 0.05 + 0$$
$$= 1$$

3. Suppose the random variables X and Y have joint density function

$$f(x, y) = \begin{cases} \frac{6}{7}(x+y)^2 & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) (5 points) Find $P(X + Y \le 1)$.



$$P(X+Y\le 1)$$
$$= \int_0^1 \int_0^{1-y} \frac{6}{7}(x+y)^2 \, dx \, dy$$
$$= \int_0^1 \frac{2}{7}(x+y)^3 \Big|_{x=0}^{1-y} dy$$
$$= \int_0^1 \frac{2}{7}\left[1 - y^3\right] dy$$
$$= \frac{2}{7}\left[1 - \frac{1}{4}\right]$$
$$= \frac{3}{14}$$

(b) (5 points) Find the marginal density of Y.

$$f_Y(y) = \int_0^1 \frac{6}{7}(x+y)^2\,dx$$

$$= \frac{2}{7}\left[(1+y)^3 - y^3\right], \quad 0 \le y \le 1$$

and 0 otherwise

(c) (3 points) Are X and Y independent? Explain.

No

Density cannot be factored.

4. (7 points) X_1, X_2, and X_3 are independent random variables with variances $\mathrm{Var}(X_1) = 1$, $\mathrm{Var}(X_2) = 4$, and $\mathrm{Var}(X_3) = 9$. Find the correlation between $Y = X_1 - X_2$ and $Z = X_2 + X_3$.

$$V(Y) = V(X_1) + V(X_2) = 5$$

$$V(Z) = V(X_2) + V(X_3) = 13$$

since independent

$$\mathrm{Cov}(Y, Z) = \mathrm{Cov}(X_1 - X_2, X_2 + X_3)$$

$$= \mathrm{Cov}(X_1, X_2) + \mathrm{Cov}(X_2, X_3) - \mathrm{Cov}(X_2, X_3) - \mathrm{Cov}(X_2, X_2)$$

$$= -V(X_2) = -4$$

$$\rho(Y, Z) = \frac{-4}{\sqrt{5}\sqrt{13}}$$

5. (5 points) A machine used to fill cereal boxes dispenses, on average, μ grams per box. The manufacturer wants the actual grams dispensed, X, to be within two grams of μ at least 75% of the time. What is the largest value of σ^2, the variance of X, that can be tolerated if the manufacturer's objectives are to be met?

Want $P(|X-\mu| > 2) \leq 0.25$

By Chebyshev's Inequality

$$P(|X-\mu| > 2) \leq \frac{V(X)}{2^2} = \frac{\sigma^2}{4}$$

So require

$$\frac{\sigma^2}{4} \leq 0.25$$

$$\Rightarrow \sigma^2 \leq 1$$