STA 257S - Summer, 1996
Test #1
July 15, 1996

INSTRUCTIONS:

- Time: 50 minutes
- No aids allowed.
- Answers that are algebraic expressions should be simplified. Series and integrals should be evaluated. Numerical answers need not be expressed in decimal form.
- Total points: 50

NAME: SOLUTIONS.

STUDENT NUMBER:

TUTOR:

1. The following questions involve a coin being tossed repeatedly. Assume that, on each toss, there are two equally likely outcomes (Heads and Tails).

   (a) (6 points) The coin is tossed three times. The outcome of every toss is of interest. Describe the probability space.

S - sample space (set of possible outcomes)

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

$\mathcal{F}$ - event space.
- set of all possible subsets of S (2^8 elements)

P - probability measure
$P(w) = \frac{1}{8}$ for all $w \in S$

(b) (6 points) Let the random variable X be the number of Heads minus the number of Tails in the first four tosses of the coin. What is the probability mass function for X?

Possibel values for X: $\{-4, -2, 0, 2, 4\}$

$P(X=4) = P(X=-4) = \frac{1}{2^4}$

$P(X=2) = P(X=-2) = \binom{4}{1}\frac{1}{2^4}$

$P(X=0) = \binom{4}{2}\frac{1}{2^4}$

(c) Suppose the coin is tossed until, for the first time, the same result appears two times in succession. Let the random variable Y be the total number of tosses required.

i. (3 points) What is the probability mass function for Y?

$P(Y=y) = \frac{1}{2^{y-1}}$, $y = 2, 3, 4, \ldots$
(0 otherwise)

ii. (5 points) Find the probability that an odd number of tosses is required.

$P(Y \text{ odd}) = P(Y=3) + P(Y=5) + P(Y=7) + \cdots$

$= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots$

$= \frac{\frac{1}{4}}{1-\frac{1}{4}}$

$= \frac{1}{3}$

2. (5 points) A random variable X takes the values $0, 1, 2, \ldots$ with positive probability and $P(X \geq k) = \left(\frac{1}{3}\right)^k$ for $k = 0, 1, 2, \ldots$ Identify the distribution of X, including the value of any parameters.

$$P(X=k) = P(X \geq k) - P(X \geq k+1)$$
$$= \left(\frac{1}{3}\right)^k - \left(\frac{1}{3}\right)^{k+1}$$
$$= \left(\frac{1}{3}\right)^k \left(1 - \frac{1}{3}\right)$$
$$= \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)$$

X has a Geometric distribution with parameter $\frac{2}{3}$

3. (7 points) $X \sim \textit{Exponential}(\lambda = 2)$. Give the density function of X and find $P(2X^2 + 5 > 55)$.

Density: $f(x) = 2e^{-2x}$ for $x > 0$ (0 otherwise)

$$P(2X^2 + 5 > 55) = P(X > 5)$$
$$= 1 - \int_0^5 2e^{-2x}\,dx$$
$$= e^{-10}$$

4. (6 points) Suppose $X \sim \textit{Poisson}(\lambda)$ (*i.e.* X has probability mass function $p(x) = \frac{\lambda^x e^{-\lambda}}{x!}$ for $x = 0, 1, 2, 3, \ldots$) Find $E(\frac{1}{1+X})$.

$$E\left(\frac{1}{1+X}\right) = \sum_{x=0}^{\infty} \frac{1}{1+x} \frac{\lambda^x e^{-\lambda}}{x!}$$
$$= \frac{1}{\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x+1} e^{-\lambda}}{(x+1)!}$$
$$= \frac{1}{\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!}$$
$$= \frac{1}{\lambda}\left[1 - e^{-\lambda}\right]$$

5. Suppose that the density function for the length L of a telephone call is

$$f(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

The cost of a call is

$$C(L) = 2 + 3L \quad \text{if } L > 0$$

Find (you may **not** assume the values for the expectation and variance of an exponential random variable developed in class are known):

(a) (6 points) the mean cost of a call

$$E(L) = \int_0^\infty x e^{-x}\,dx$$

$$= -xe^{-x}\Big|_0^\infty + \int_0^\infty e^{-x}\,dx$$

$$= 1$$

Mean cost: $E(C) = E(2+3L)$

$$= 2 + 3(1) = 5$$

(b) (6 points) the variance of the cost of a call

$$E(L^2) = \int_0^\infty x^2 e^{-x}\,dx$$

$$= -2xe^{-x}\Big|_0^\infty + 2\underbrace{\int_0^\infty x e^{-x}\,dx}_{=1 \text{ from (a)}}$$

$$= 2$$

So $V(L) = 2 - 1 = 1$

Var of cost $= V(2+3L) = 9\,V(L) = 9$