UNIVERSITY OF TORONTO

Faculty of Arts and Science

AUGUST EXAMINATIONS 1996

STA 257S

Duration - 3 hours

NO AIDS ALLOWED

(Questions: 15; Pages: 10; Total Points: 115)

Answer all questions in the space provided. Show all of your work.

NAME: SOLUTIONS

STUDENT NUMBER:

TUTOR:

1. (5 points) Suppose there are three cabinets labelled A, B, and C, each of which has two drawers. Each drawer contains one coin. In cabinet A there are two gold coins, in cabinet B there are two silver coins, and in cabinet C there is one gold and one silver coin. A cabinet is chosen at random, one of the drawers is opened, and a silver coin is found. What is the probability that the other drawer in that cabinet contains a silver coin?

Let A, B, C be the events that cabinets A, B, C resp. were chosen
S be the event discovered coin is silver.

Want $P(B|S)$

By Bayes' Theorem:

$$P(B|S) = \frac{P(S|B)P(B)}{P(S|A)P(A) + P(S|B)P(B) + P(S|C)P(C)}$$

$$= \frac{1 \cdot \frac{1}{3}}{0 + 1 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3}}$$

$$= \frac{2}{3}$$

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2. (10 points) Prove each of the following.

(a) For any two events A and B, $P(A \cap B) \geq 1 - P(A^c) - P(B^c)$.

Know: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

so $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$= 1 - P(A^c) + 1 - P(B^c) - P(A \cup B)$

$= 1 - P(A^c) - P(B^c) + 1 - P(A \cup B)$

$\geq 1 - P(A^c) - P(B^c)$ since $1 - P(A \cup B) \geq 0$ since $0 \leq P(A \cup B) \leq 1$.

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(b) If $P(B|A^c) = P(B|A)$, then A and B are independent.

If $P(B|A^c) = P(B|A)$

then $\dfrac{P(B \cap A^c)}{P(A^c)} = \dfrac{P(B \cap A)}{P(A)}$

so $P(A)P(B \cap A^c) = P(B \cap A)(1 - P(A))$

$P(A)[P(B \cap A^c) + P(B \cap A)] = P(B \cap A)$

so $P(A)P(B) = P(B \cap A)$ (L.T.P.)

So A, B are indept.

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3. (15 points) Let X be a discrete random variable with probability mass function $P(X=n) = (1-\theta)\theta^{n-1}$, $n = 1, 2, 3, \ldots$, $(0 < \theta < 1)$.

(a) Show $P(X > n) = \theta^n$.

$$P(X>n) = \sum_{k=n+1}^{\infty} (1-\theta)\theta^{k-1} = (1-\theta)\theta^n \sum_{j=0}^{\infty} \theta^j$$

$$= \frac{(1-\theta)\theta^n}{1-\theta} = \theta^n.$$

(b) Show $P(X > m+n | X > m) = P(X > n)$.

$$P(X>m+n \mid X>m) = \frac{P(X>m+n,\, X>m)}{P(X>m)} = \frac{P(X>m+n)}{P(X>m)}$$

$$= \frac{\theta^{m+n}}{\theta^m} \quad \text{from (a)}$$

$$= \theta^n = P(X>n)$$

(c) Let X_1 and X_2 be independent discrete random variables with probability mass functions

$$P(X_1 = n) = (1-\theta_1)\theta_1^{n-1}, \quad n = 1, 2, 3, \ldots \ (0 < \theta_1 < 1)$$

$$P(X_2 = n) = (1-\theta_2)\theta_2^{n-1}, \quad n = 1, 2, 3, \ldots \ (0 < \theta_2 < 1)$$

Show $P(X_2 > X_1) = \frac{\theta_2(1-\theta_1)}{1-\theta_1\theta_2}$.

LTP: $P(X_2 > X_1) = \sum_{n=1}^{\infty} P(X_2 > n \mid X_1 = n) \cdot P(X_1 = n)$

$$= \sum_{n=1}^{\infty} \theta_2^n (1-\theta_1)\theta_1^{n-1}$$

$$= \theta_2 (1-\theta_1) \sum_{k=0}^{\infty} (\theta_1\theta_2)^k$$

$$= \frac{\theta_2(1-\theta_1)}{1-\theta_1\theta_2}$$

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4. (4 points) A random variable X' is said to be obtained from the random variable X by "truncation at the point a" if X' is defined by

$$X'(\omega) = \begin{cases} X(\omega) & \text{if } X(\omega) \le a \\ a & \text{if } X(\omega) > a \end{cases}$$

Express the distribution function of X' in terms of the distribution function of X.

$$F_{X'}(x) = P(X' \le x)$$

$$= \begin{cases} F_X(x) & , X(\omega) \le a \\ 1 & , X(\omega) > a \end{cases}$$

5. (7 points) X and Y are jointly distributed discrete random variables with joint mass function given in the table:

		X		
		0	3	6
Y	1	?	?	?
	2	.1	.05	?

Using the information that $P(Y=2|X=0) = \frac{1}{4}$ and that X and Y are independent, fill in the missing information in the table.

$P(Y=2) = 0.25$ (since indept.)

so $\boxed{p(6,2) = 0.1}$

Since indept, $p(6,2) = p_X(6)p_Y(2)$

$.1 = p_X(6)(.25)$

so $p_X(6) = .4$

so $\boxed{p(6,1) = 0.3}$

Similarly $p(3,2) = p_X(3)p_Y(2)$

$.05 = p_X(3)(.25)$

$p_X(3) = 0.2$, $\boxed{\text{so } p(3,1) = .15}$

By subtraction $\boxed{p(0,1) = .3}$

$\frac{.05}{.25} = \frac{5}{25} = \frac{1}{5}$

y \ x	0	3	6
1	.3	.15	.3
2	.1	.05	.1
		.2	.4

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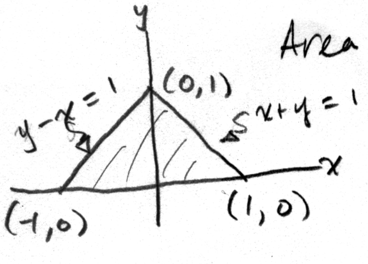
6. (17 points) Suppose X and Y are uniformly distributed over the triangle with vertices $(-1,0)$, $(0,1)$, and $(1,0)$.

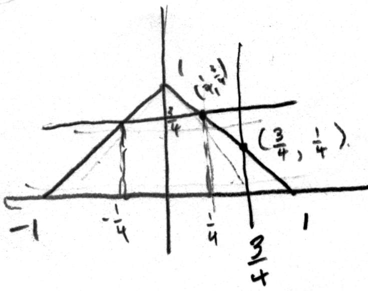
(a) What is the joint density function of X and Y?



Area = 1

$$\underline{\text{density}}: \ f_{X,Y}(x,y) = \begin{cases} 1 & y+|x| \le 1, \ 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(b) Find $P(X \le \frac{3}{4}, Y \le \frac{3}{4})$.



$$= 1 - \underbrace{\tfrac{1}{2}\left(\tfrac{1}{2}\right)\left(\tfrac{1}{4}\right)}_{\text{area of top triangle}} - \underbrace{\tfrac{1}{2}\left(\tfrac{1}{4}\right)\left(\tfrac{1}{4}\right)}_{\text{area of right triangle}}$$

$$= 1 - \frac{1}{16} - \frac{1}{32} = \frac{29}{32}$$

(c) Find $E(XY)$.

$$E(XY) = \int_{-1}^{0}\int_{0}^{1+x} xy \, dy \, dx + \int_{0}^{1}\int_{0}^{1-x} xy \, dy \, dx$$

$$= \int_{-1}^{0} \frac{xy^2}{2}\Big|_{y=0}^{1+x} dx + \int_{0}^{1} \frac{xy^2}{2}\Big|_{y=0}^{1-x} dx$$

$$= \int_{-1}^{0} \frac{x(1+x)^2}{2} dx + \int_{0}^{1} \frac{x(1-x)^2}{2} dx$$

$$= \frac{1}{2}\int_{-1}^{0} (x + 2x^2 + x^3)\,dx + \frac{1}{2}\int_{0}^{1} (x - 2x^2 + x^3)\,dx$$

$$= \frac{1}{2}\left\{\left[\frac{x^2}{2} + \frac{2}{3}x^3 + \frac{x^4}{4}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{2}{3}x^3 + \frac{x^4}{4}\right]_{0}^{1}\right\}$$

$$= \frac{1}{2}\left\{-\frac{1}{2} + \frac{2}{3} - \frac{1}{4} + \frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right\}$$

$$= 0$$

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7. (5 points) Suppose X and Y are discrete random variables with joint probability mass function $p(x, y)$. Prove $E(aX + bY) = aE(X) + bE(Y)$, where $a, b \in \mathbf{R}$.

$$E(aX+bY) = \sum_x \sum_y (ax+by)\, p(x,y)$$

$$= a \sum_x x \sum_y p(x,y) + b \sum_y y \sum_x p(x,y)$$

$$= a \sum_x x p_X(x) + b \sum_y y p_Y(y)$$

$$= aE(X) + bE(Y)$$

8. (5 points) If $X \sim \text{Unif}(0, 1)$ find the density of $Y = -2\log(X)$.

Let $h(x) = -2\log x \Rightarrow$

Since $h(x)$ is monotone, can apply the thm.

$$h^{-1}(y) = e^{-y/2}$$

$$\frac{d}{dy} h^{-1}(y) = -\frac{1}{2} e^{-y/2}$$

$$f_Y(y) = \begin{cases} \frac{1}{2} e^{-y/2} & , y > 0 \\ 0 & , \text{otherwise.} \end{cases}$$

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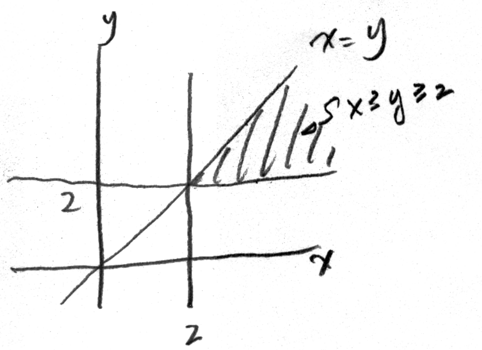
9. (15 points) Suppose X and Y are independent, identically distributed exponential random variables with parameter λ.

(a) Find $P(X \geq Y \geq 2)$.

$f_{X,Y}(x,y) = \lambda^2 e^{-\lambda x} e^{-\lambda y}$, $x, y > 0$



$$P(X \geq Y \geq 2) = \int_2^\infty \int_2^x \lambda^2 e^{-\lambda x} e^{-\lambda y} \, dy \, dx$$

$$= \int_2^\infty -\lambda e^{-\lambda x} e^{-\lambda y} \Big|_{y=2}^{x} dx$$

$$= \int_2^\infty \left(\lambda e^{-2\lambda} e^{-\lambda x} - \lambda e^{-2\lambda x} \right) dx$$

$$= -e^{-2\lambda} e^{-\lambda x} \Big|_{x=2}^\infty + \frac{1}{2} e^{-2\lambda x} \Big|_{x=2}^\infty = \frac{1}{2} e^{-4\lambda}$$

(b) Find the joint density function of $U = \frac{X}{Y}$ and $V = X + Y$. Are U and V independent?

$x = uy = u(v-x)$, $y = v - \frac{uv}{1+u} = \frac{v}{1+u}$

so $x = \frac{uv}{1+u}$

$$\frac{\partial x}{\partial u} = \frac{v(1+u) - uv}{(1+u)^2} = \frac{v}{(1+u)^2} \qquad \frac{\partial y}{\partial u} = -\frac{v}{(1+u)^2}$$

JACOBIAN
$$\begin{vmatrix} \frac{v}{(1+u)^2} & \frac{u}{1+u} \\ -\frac{v}{(1+u)^2} & \frac{1}{1+u} \end{vmatrix} = \frac{v}{(1+u)^3} + \frac{uv}{(1+u)^3} = \frac{v}{(1+u)^2}$$

$$f_{U,V}(u,v) = \begin{cases} \lambda^2 e^{-\lambda v} \dfrac{v}{(1+u)^2}, & u, v > 0 \\ 0, & \text{otherwise.} \end{cases}$$

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10. (3 points) Suppose T has a t distribution with n degrees of freedom. What is the distribution of T^2? State the value of any parameters.

$$T^2 \sim F(1, n).$$

11. (8 points) Use probability generating functions to find the distribution of $Y = X_1 + X_2$ where X_1 and X_2 are Poisson random variables with parameters λ_1 and λ_2, respectively. (The probability mass function for the Poisson distribution is $p(k) = \frac{\lambda^k e^{-\lambda}}{k!}$, $k = 0, 1, 2, \ldots$)

p.g.f for X_1

$$\pi(t) = \sum_{k=0}^{\infty} \frac{\lambda_1^k e^{-\lambda_1}}{k!} t^k = e^{-\lambda_1} e^{\lambda_1 t} = e^{-\lambda_1(1-t)}.$$

p.g.f for $X_1 + X_2$:

$$\pi_{X_1+X_2}(t) = \pi_{X_1}(t)\,\pi_{X_2}(t)$$

$$= e^{-\lambda_1(1-t)} e^{-\lambda_2(1-t)}$$

$$= e^{-(\lambda_1+\lambda_2)(1-t)}$$

which is the p.g.f. for a Poisson r.v. with parameter $\lambda_1 + \lambda_2$

So $Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$

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12. (6 points) For each of the following functions, state whether or not it is a moment generating function. If not, explain why not. If so, find the underlying distribution.

(a) $m(t) = \frac{e^t}{4-e^t}$

$m(0) = \frac{1}{3} \neq 1$

so NOT a m.g.f.

(b) $m(t) = \frac{3e^{4t}+e^{-2t}}{4}$ $(m(0)=1)$

Since $m(t) = E(e^{tX})$ this is the mgf for a discrete r.v. X whose mass function is $p(4) = \frac{3}{4}$, $p(-2) = \frac{1}{4}$.

13. (5 points) Let $m(t)$ be the moment generating function of the random variable X and define $\kappa(t) = \log m(t)$. Show that

$$\frac{d}{dt}\kappa(t)\bigg|_{t=0} = E(X)$$

and

$$\frac{d^2}{dt^2}\kappa(t)\bigg|_{t=0} = \mathrm{Var}(X).$$

$$\frac{d}{dt}K(t)\bigg|_{t=0} = \frac{1}{m(t)}m'(t)\bigg|_{t=0} = E(X)$$ since $m'(0) = EX$, $m(0) = 1$

$$\frac{d^2}{dt^2}K(t)\bigg|_{t=0} = \frac{m(t)m''(t) - [m'(t)]^2}{[m(t)]^2}\bigg|_{t=0}$$

$= E(X^2) - [E(X)]^2$ since $m''(0) = E(X^2)$

$= \mathrm{Var}(X)$

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14. (5 points) Let X be a non-negative random variable such that its moment generating function, $m(t)$, is finite for all t. Prove $P(X \geq a) \leq e^{-ta}m(t)$ for $t \geq 0$, and a a positive constant.

$$P(X \geq a) = P(tX \geq ta) \quad \text{for } t \geq 0$$

$$= P(e^{tX} \geq e^{ta}) \quad \text{since } e^{x} \text{ is monotone incr.}$$

$$\leq \frac{E(e^{tX})}{e^{ta}} \quad \text{by Chebyshev's Inequality}$$

$$= e^{-ta} m(t)$$

//.

15. (5 points) A fair die is rolled 12000 times. Let S be the total number of sixes. Use the Central Limit Theorem to find $P(1900 < S < 2200)$ in terms of Φ, the standard normal distribution function.

$$E(S) = \frac{12000}{6} = 2000$$

$$Var(S) = 12000\left(\tfrac{1}{6}\right)\left(\tfrac{5}{6}\right) = \frac{5000}{3}$$

by CLT, $S \approx N\left(2000, \frac{5000}{3}\right)$

$$P(1900 < S < 2200)$$

$$= P(S < 2200) - P(S < 1900)$$

$$\approx \Phi\left(\frac{2200-2000}{\sqrt{5000/3}}\right) - \Phi\left(\frac{1900-2000}{\sqrt{\frac{5000}{3}}}\right)$$

Total Pages = (10)
Total Points = (115)