UNIVERSITY OF TORONTO

Faculty of Arts and Science

AUGUST EXAMINATIONS 1996

STA 257S

Duration - 3 hours

NO AIDS ALLOWED

(Questions: 15; Pages: 10; Total Points: 115)

Answer all questions in the space provided. Show all of your work.

NAME:______________________________

STUDENT NUMBER:______________________________

TUTOR:______________________________

1. (5 points) Suppose there are three cabinets labelled A, B, and C, each of which has two drawers. Each drawer contains one coin. In cabinet A there are two gold coins, in cabinet B there are two silver coins, and in cabinet C there is one gold and one silver coin. A cabinet is chosen at random, one of the drawers is opened, and a silver coin is found. What is the probability that the other drawer in that cabinet contains a silver coin?

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[redacted]

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2. (10 points) Prove each of the following.

(a) For any two events A and B, $P(A \cap B) \geq 1 - P(A^c) - P(B^c)$.

[redacted]

(b) If $P(B|A^c) = P(B|A)$, then A and B are independent.

[redacted]

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3. (15 points) Let X be a discrete random variable with probability mass function $P(X = n) = (1 - \theta)\theta^{n-1}$, $n = 1, 2, 3, \ldots$, $(0 < \theta < 1)$.

   (a) Show $P(X > n) = \theta^n$.

   (b) Show $P(X > m + n | X > m) = P(X > n)$.

   (c) Let X_1 and X_2 be independent discrete random variables with probability mass functions

   $$P(X_1 = n) = (1 - \theta_1)\theta_1^{n-1}, \quad n = 1, 2, 3, \ldots \ (0 < \theta_1 < 1)$$

   $$P(X_2 = n) = (1 - \theta_2)\theta_2^{n-1}, \quad n = 1, 2, 3, \ldots \ (0 < \theta_2 < 1)$$

   Show $P(X_2 > X_1) = \frac{\theta_2(1-\theta_1)}{1-\theta_1\theta_2}$.

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4. (4 points) A random variable X' is said to be obtained from the random variable X by "truncation at the point a" if X' is defined by

$$X'(\omega) = \begin{cases} X(\omega) & \text{if } X(\omega) \leq a \\ a & \text{if } X(\omega) > a \end{cases}$$

Express the distribution function of X' in terms of the distribution function of X.

5. (7 points) X and Y are jointly distributed discrete random variables with joint mass function given in the table:

		X		
		0	3	6
Y	1	?	?	?
	2	.1	.05	?

Using the information that $P(Y = 2|X = 0) = \frac{1}{4}$ and that X and Y are independent, fill in the missing information in the table.

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6. (17 points) Suppose X and Y are uniformly distributed over the triangle with vertices $(-1, 0)$, $(0, 1)$, and $(1, 0)$.

   (a) What is the joint density function of X and Y?

   (b) Find $P(X \leq \frac{3}{4}, Y \leq \frac{3}{4})$.

   (c) Find $E(XY)$.

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7. (5 points) Suppose X and Y are discrete random variables with joint probability mass function $p(x, y)$. Prove $E(aX + bY) = aE(X) + bE(Y)$, where $a, b \in \mathbf{R}$.

8. (5 points) If $X \sim \text{Unif}(0, 1)$ find the density of $Y = -2\log(X)$.

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9. (15 points) Suppose X and Y are independent, identically distributed exponential random variables with parameter λ.

(a) Find $P(X \geq Y \geq 2)$.

(b) Find the joint density function of $U = \frac{X}{Y}$ and $V = X + Y$. Are U and V independent?

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10. (3 points) Suppose T has a t distribution with n degrees of freedom. What is the distribution of T^2? State the value of any parameters.

11. (8 points) Use probability generating functions to find the distribution of $Y = X_1 + X_2$ where X_1 and X_2 are independent Poisson random variables with parameters λ_1 and λ_2, respectively. (The probability mass function for the Poisson distribution is $p(k) = \frac{\lambda^k e^{-\lambda}}{k!}$, $k = 0, 1, 2, \ldots$)

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12. (6 points) For each of the following functions, state whether or not it is a moment generating function. If not, explain why not. If so, find the underlying distribution.

(a) $m(t) = \frac{e^t}{4-e^t}$

(b) $m(t) = \frac{3e^{4t}+e^{-2t}}{4}$

13. (5 points) Let $m(t)$ be the moment generating function of the random variable X and define $\kappa(t) = \log m(t)$. Show that

$$\left.\frac{d}{dt}\kappa(t)\right|_{t=0} = E(X)$$

and

$$\left.\frac{d^2}{dt^2}\kappa(t)\right|_{t=0} = \text{Var}(X).$$

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14. (5 points) Let X be a non-negative random variable such that its moment generating function, $m(t)$, is finite for all t. Prove $P(X \geq a) \leq e^{-ta}m(t)$ for $t \geq 0$, and a a positive constant.

15. (5 points) A fair die is rolled 12000 times. Let S be the total number of sixes. Use the Central Limit Theorem to find $P(1900 < S < 2200)$ in terms of Φ, the standard normal distribution function.

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