STA 303 H1S / 1002 HS – Winter 2011
Test
March 7, 2011

LAST NAME: SOLUTIONS FIRST NAME:________________

STUDENT NUMBER:________________

ENROLLED IN: (circle one) STA 303 STA 1002

INSTRUCTIONS:

- Time: 90 minutes
- Aids allowed: calculator.
- Some formulae are on the last page (page 11).
- Total points: 45

1ab	1cde	2abcde	2fghij	3

1. Many countries survey a random sample of adults every year or two to collect demographic information and opinions on issues ranging from government spending to the state of race relations to the existence and nature of God.

   In this question, we will consider data collected from 943 people aged 18-30 in the 2004 American General Social Survey. The variables we will consider are gender (male or female), political party affiliation (Democrat, Independent, or Republican), and response on a 7 point scale to a question rating their political ideology where 1=extremely liberal, 2=liberal, 3=slightly liberal, 4=moderate, 5=slightly conservative, 6=conservative, and 7=extremely conservative.

   For this question, our interest is in how the levels of liberalism or conservatism in the responses to the political ideology question vary with political party affiliation and gender. Thus we will treat political ideology level as a quantitative variable (variable name: `ideology`) and examine how its mean differs among the groups of subjects, categorized by political party affiliation (variable name: `party`) and gender (variable name: `gender`). The higher the mean ideology, the more conservative the group's responses tended to be.

   Some edited SAS output from 2 models is below and on the next page. Some numbers have been replaced by `X`'s.

MODEL 1

```
                               The GLM Procedure

                           Class Level Information

           Class         Levels    Values
           party              3    Democrat Independent Republican
           gender             2    Female Male

                    Number of Observations Read         943
                    Number of Observations Used         943

Dependent Variable: ideology

                                      Sum of
Source                     DF        Squares    Mean Square   F Value   Pr > F
Model                       5      90.332492      18.066498     10.81   <.0001
Error                     937    1565.885959       1.671170
Corrected Total           942    1656.218452

             R-Square     Coeff Var      Root MSE    ideology Mean
             0.054541      31.55711      1.292737         4.096501

Source                     DF    Type III SS    Mean Square   F Value   Pr > F
party                       2    87.79540434    43.89770217     26.27   <.0001
gender                      1     1.48829759     1.48829759      0.89   0.3456
party*gender                X     XXXXXXXXXX     XXXXXXXXXX      XXXX   0.3370
```

(SAS output for question 1 continues on the next page)

(SAS output for question 1 continued)

```
                                                            Standard
Parameter                             Estimate                 Error    t Value    Pr > |t|
Intercept                          4.664062500 B          0.11426291      40.82      <.0001
party        Democrat             -0.896062500 B          0.16255882      -5.51      <.0001
party        Independent          -0.622395833 B          0.15703935      -3.96      <.0001
party        Republican            0.000000000 B           .                 .           .
gender       Female               -0.231963735 B          0.15287863      -1.52      0.1295
gender       Male                  0.000000000 B           .                 .           .
party*gender Democrat Female       0.310475362 B          0.21098368       1.47      0.1415
party*gender Democrat Male         0.000000000 B           .                 .           .
party*gender Independent Female    0.142959790 B          0.21181548       0.67      0.4999
party*gender Independent Male      0.000000000 B           .                 .           .
party*gender Republican Female     0.000000000 B           .                 .           .
party*gender Republican Male       0.000000000 B           .                 .           .

NOTE: The X'X matrix has been found to be singular, and a generalized inverse was used to
      solve the normal equations.  Terms whose estimates are followed by the letter 'B'
      are not uniquely estimable.
```

MODEL 2

(Initial output that is the same as for MODEL 1 has been deleted)

```
Dependent Variable: ideology

                                        Sum of
Source                      DF         Squares    Mean Square    F Value    Pr > F
Model                        3       86.692970      28.897657      17.29    <.0001
Error                      939     1569.525482       1.671486
Corrected Total            942     1656.218452

             R-Square     Coeff Var      Root MSE    ideology Mean
             0.052344      31.56010      1.292860         4.096501

Source                      DF     Type III SS    Mean Square    F Value    Pr > F
party                        2     84.25160525    42.12580263      25.20    <.0001
gender                       1      1.31102845     1.31102845       0.78    0.3760

                                                  Standard
Parameter                   Estimate                 Error    t Value    Pr > |t|
Intercept                4.576815040 B          0.08971327      51.02      <.0001
party     Democrat      -0.711248269 B          0.10353591      -6.87      <.0001
party     Independent   -0.542288453 B          0.10538744      -5.15      <.0001
party     Republican     0.000000000 B           .                 .           .
gender    Female        -0.075780010 B          0.08556576      -0.89      0.3760
gender    Male           0.000000000 B           .                 .           .
```

(Questions related to this output begin on the next page.)

(a) (4 marks) Write the **model** that is being estimated in the output labelled MODEL 1; define all variables.

$$Y \quad = \quad \beta_0 + \beta_1 I_{[Dem]} + \beta_2 I_{[Ind]} + \beta_3 I_{[Female]} + \beta_4 I_{[Dem]} * I_{[Female]} + \beta_5 I_{[Ind]} * I_{[Female]} + e$$

where:
Y *is the political ideology score,*
$I_{[Dem]}$ *is 1 if party affiliation is Democrat and 0 otherwise,*
$I_{[Ind]}$ *is 1 if party affiliation is Independent and 0 otherwise,*
$I_{[Female]}$ *is 1 if Female and 0 otherwise, and*
e *is random error*

(b) For the test in MODEL 1 with p-value 0.3370:

i. (1 mark) What are the null and alternative hypotheses?

$H_0 : \quad \beta_4 = \beta_5 = 0$
H_a : *at least one of* β_4, β_5 *is not 0*

ii. (2 marks) Explain in *practical* terms what you conclude from the test.

There is no evidence that differences in mean political ideology scores among party affiliations differ with gender.

iii. (4 marks) What are the 4 missing numbers?

`DF` = 2

`Type III SS` = $1569.525 - 1565.886 = 3.64$

`Mean Square` = $3.64/2 = 1.82$

`F Value` = $1.82/1.67 = 1.09$

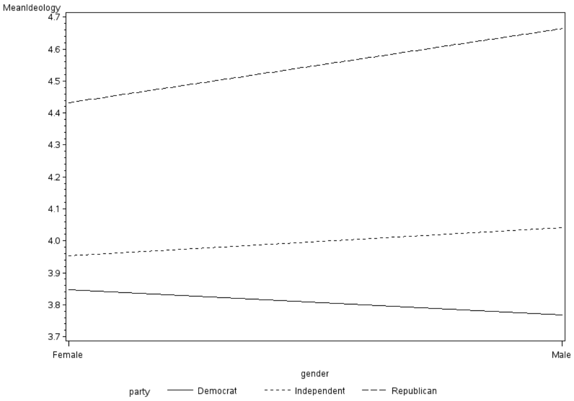
(c) (1 mark) For MODEL 2, what practical quantity, if anything, is being estimated by the estimate of the intercept?

The mean political ideology for male Republicans.

(d) (1 mark) For MODEL 2, estimate the mean political ideology score for females whose party affiliation is Democrat.

$$4.577 - 0.711 - 0.076 = 3.79$$

(e) (6 marks) Here is a plot showing the mean value of political ideology for each gender, with separate lines for each party affiliation. (Females are on the left and males are on the right. The top line is for Republicans, middle line is for Independents, and bottom line is for Democrats.)



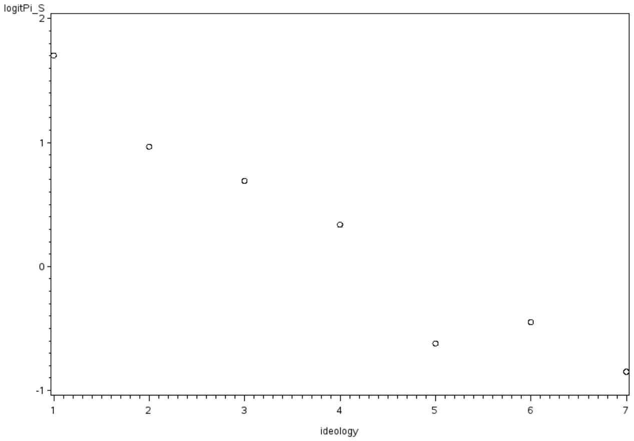
Explain how the interaction plot is consistent with the conclusions that can be drawn from inferences about the fitted model. Support your answer with relevant numbers from the SAS output.

Inference	*Plot*	*Relevant p-value*
There is no evidence of an interaction	*Lines are close to parallel*	*0.3370*
There is no evidence of a difference between genders	*Lines are close to horizontal*	*0.3760*
There is a strong evidence of a difference among political party affiliations	*Line for Republicans is much higher than lines for other party affilitions*	*<0.0001*

2. In this question, we will work with the same data as in question 1. However, we will only consider people whose party affiliation is Democrat or Republican (people who identified themselves as Independent have been removed from the data) and we will ignore gender. Here we will consider a model for how well party affiliation can be predicted from a person's political ideology score.

   Of the m_i people who responded that their political ideology is level i ($i = 1, \ldots, 7$), y_i (variable name: `nDemocrats`) is the number whose party affiliation is Democrat. Political ideology (variable name: `ideology`) is treated as a quantitative explanatory variable in this analysis.

   Some SAS output is given below and on the next page. The plot is the logit of the response proportions versus political ideology score. A few numbers in the output have been replaced by `X`'s.



```
          The LOGISTIC Procedure
                Model Information
Data Set                        WORK.BINOMIAL
Response Variable (Events)      nDemocrats
Response Variable (Trials)      m
Model                           binary logit
Optimization Technique          Fisher's scoring

    Number of Observations Read           7
    Number of Observations Used           7
    Sum of Frequencies Read             630
    Sum of Frequencies Used             630

                Response Profile
       Ordered     Binary           Total
         Value     Outcome      Frequency
             1     Event              340
             2     Nonevent           290
```

(SAS output for this question continues on the next page.)

(SAS output for question 2 continued)

```
                    Model Convergence Status
         Convergence criterion (GCONV=1E-8) satisfied.

         Deviance and Pearson Goodness-of-Fit Statistics

  Criterion          Value       DF     Value/DF     Pr > ChiSq
  Deviance          6.0544        5       1.2109         0.3010
  Pearson           5.9719        5       1.1944         0.3090

             Number of events/trials observations: 7

                     Model Fit Statistics
                                         Intercept
                          Intercept            and
             Criterion         Only     Covariates
             AIC            871.393        827.862
             SC             875.839        836.753
             -2 Log L       869.393        823.862

             Testing Global Null Hypothesis: BETA=0
     Test                 Chi-Square       DF     Pr > ChiSq
     Likelihood Ratio        XXXXXXX        1         <.0001
     Score                   XXXXXXX        1         <.0001
     Wald                    XXXXXXX        1         <.0001

            Analysis of Maximum Likelihood Estimates
                                 Standard          Wald
Parameter    DF    Estimate       Error     Chi-Square    Pr > ChiSq
Intercept     1      1.9110      0.2864        44.5265        <.0001
ideology      1     -0.4194      0.0652        41.3812        <.0001

                     Odds Ratio Estimates
                        Point          95% Wald
           Effect      Estimate      Confidence Limits
           ideology       XXXXX      XXXXX       XXXXX


                                    n         Pearson
    Obs    ideology     m      Democrats        Res        DevRes

     1         1        13         11        0.27773     0.28405
     2         2        69         50       -0.38861    -0.38534
     3         3        90         60        0.18010     0.18049
     4         4       238        139        0.80536     0.80719
     5         5       100         35       -2.08213    -2.10377
     6         6       100         39        0.77133     0.76572
     7         7        20          6        0.36400     0.35908
```

(Questions related to this output begin on the next page.)

(a) (2 marks) On page 6 you are given a plot of the logit of the response proportions versus political ideology score. What is the purpose of looking at this plot and what do you conclude from it?

Look at it to see if a linear relationship in political ideology seems to be appropriate. It appears to be linear, although the value when political ideology is 5 seems not to follow the pattern well.

(b) (2 marks) Why is the "`Number of Observations Read`" 7? And how is the "`Sum of Frequencies Read`" arrived at?

There is one observation for each value of political ideology which gives the 7.
The sum of the frequencies is the total of the numbers of people who responded to each level of political ideology.

(c) (3 marks) Write the log-likelihood function in terms of y_i, m_i, and the model parameters to be estimated.

Likelihood function:

$$L = \prod_{i=1}^{7} \left(\begin{array}{c} m_i \\ y_i \end{array} \right) \pi_i^{y_i} (1-\pi_i)^{(m_i - y_i)}$$

Log-likelihood function:

$$\log(L) = \sum_{i=1}^{7} \left\{ \log \left(\begin{array}{c} m_i \\ y_i \end{array} \right) + y_i \log(\pi_i) + (m_i - y_i) \log(1-\pi_i) \right\}$$

where

$$\pi_i = \frac{\exp(\beta_0 + \beta_1 x_i)}{1 + \exp(\beta_0 + \beta_1 x_i)}$$

and x_i is the political ideology for the ith observation.

(d) (2 marks) What is the fitted equation? Define all variables.

$$\log\left(\frac{\hat{\pi}_i}{1-\hat{\pi}_i}\right) = 1.91 - 0.42\,\texttt{ideology}_i$$

where $\texttt{ideology}_i$ is the political ideology score for the ith observation and π_i is the probability that a person with that political ideology score is a Democrat.

(e) (1 mark) How would your answer to part (d) change if y_i was changed to the number of people whose party affiliation was Republican?

The right side of the equation would be $-1.91 + 0.42\,\texttt{ideology}_i$.

(f) (1 mark) From the model, what is the estimate of the probability of belonging to the Democratic party for someone whose response to the political ideology question is 4?

$$\frac{\exp(1.91 - 0.42(4))}{1 + \exp(1.91 - 0.42(4))} = 0.56$$

(g) (3 marks) What is the odds ratio estimate for the effect of ideology? For higher responses on the political ideology scale, are the odds of being a Democrat higher or lower? By how much?

Odds ratio: $\exp(-0.42) = 0.66$
For higher responses, the odds of being a Democrat are lower.
For each point higher the response on the political ideology scale, the odds of being a Democrat are 66% of what they were for the lower response.

(h) (1 mark) Write down the saturated model.

$$\begin{aligned} \text{logit}(\pi) &= \beta_0 + \beta_1 I_{[ideology=1]} + \beta_2 I_{[ideology=2]} + \beta_3 I_{[ideology=3]} + \beta_4 I_{[ideology=4]} \\ &\quad + \beta_5 I_{[ideology=5]} + \beta_6 I_{[ideology=6]} \end{aligned}$$

(i) (4 marks) From the SAS output that you are given, can you carry out a Likelihood Ratio Test to test whether the coefficient of ideology is statistically significantly different from 0? If yes, carry it out, giving each of the following: (I) the test statistic, (II) the distribution of the test statistic under the null hypothesis, (III) the p-value, (IV) the conclusion. If no, state what is needed to find each of these 4 things.

YES!
Test statistic: $869.393 - 823.862 = 45.531$
Distribution of the test statistic under H_0*: chi-square with 1 df*
p-value: < 0.0001
Conclusion: There is strong evidence that the coefficient is not 0, so the odds of being a Democrat are related to the response to the political ideology question.

(j) (1 mark) You are given the Pearson (`PearsonRes`) and Deviance (`DevRes`) residuals? What can you conclude from them?

Both residuals are much smaller for the 5th observation than for all the other observations; it can be considered an outlier. The model overestimates the log-odds of being a Democrat for a person whose response to the political ideology question is 5.

3. (6 marks) Fill out the table below to compare and contrast features of the models used in questions 1 and 2.

	First Model in Question 1	Model in Question 2
Underlying probability distribution of the response (You do not need to specify the parameters of the distribution.)	*Normal*	*Binomial*
Condition that must hold regarding the variance in order for inferences to be valid	*Same for all observations*	*Variance varies for each observation with the value of* π_i *and* m_i*; the variance is* $m_i\pi_i(1-\pi_i)$
Probability distribution used to calculate the p-value for the test with null hypothesis that the coefficients of all parameters except the intercept are 0 (You do not need to specify the parameters (or df) of the distribution.)	F	*chi-square*

Some formulae:

Pooled t-test

$$t_{obs} = \frac{\overline{y}_1 - \overline{y}_2}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

Linear Regression

$$b_1 = \frac{\sum (x_i - \overline{x})(y_i - \overline{y})}{\sum (x_i - \overline{x})^2} = \frac{\sum x_i y_i - n\overline{x}\overline{y}}{\sum x_i^2 - n\overline{x}^2}$$

$$b_0 = \overline{y} - b_1\overline{x}$$

One-way analysis of variance

$$\text{SSTO} = \sum_{i=1}^{N} (y_i - \overline{y})^2$$

$$\text{SSE} = \sum_{g=1}^{G} \sum_{(g)} (y_i - \overline{y}_g)^2$$

$$\text{SSR} = \sum_{g=1}^{G} n_g (\overline{y}_g - \overline{y})^2$$

Bernoulli and Binomial distributions

If $Y \sim \text{Bernoulli}(\pi)$
$\text{E}(Y) = \pi$, $\text{Var}(Y) = \pi(1 - \pi)$

If $Y \sim \text{Binomial}(m, \pi)$
$\text{E}(Y) = m\pi$, $\text{Var}(Y) = m\pi(1 - \pi)$

Logistic Regression with Binomial Response formulae

$$\text{Deviance} = 2\sum_{i=1}^{n} \{y_i \log(y_i) + (m_i - y_i)\log(m_i - y_1) - y_i \log(\hat{y}_i) + (m_i - y_i)\log(m_i - \hat{y}_1)\}$$

$$D_{res,i} = \text{sign}(y_i - m_i\hat{\pi}_i)\sqrt{2\left\{y_i \log\left(\frac{y_i}{m_i\hat{\pi}_i}\right) + (m_i - y_i)\log\left(\frac{m_i - y_i}{m_i - m_i\hat{\pi}_i}\right)\right\}}$$

$$P_{res,i} = \frac{y_i - m_i\hat{\pi}_i}{\sqrt{m_i\hat{\pi}_i(1 - \hat{\pi}_i)}}$$

Model Fitting Criteria

$$\text{AIC} = -2\log(L) + 2(p + 1)$$

$$\text{SC} = -2\log(L) + (p + 1)\log(N)$$