UNIVERSITY OF TORONTO

Faculty of Arts and Science

DECEMBER EXAMINATIONS 2005

STA 302 H1F / STA 1001 H1F

Duration - 3 hours

Aids Allowed: Calculator

LAST NAME:________________________**FIRST NAME:**______________________

STUDENT NUMBER: ____

- There are 17 pages including this page.
- The last page is a table of formulae that may be useful. For all questions you can assume that the results on the formula page are known.
- Tables of the t distribution can be found on page 14 and tables of the F distribution can be found on pages 15 and 16.
- Total marks: 95

1	2abc	2de	2f	3	4ab

4cde	5a	5bcde	6	7

1. Suppose we have $n = 102$ pairs $(X_1, Y_1), \ldots, (X_n, Y_n)$ and we fit the simple linear regression model $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$. Here are some summary statistics:

$$\overline{X} = 50 \qquad \overline{Y} = 100$$
$$\sum_{i=1}^n (X_i - \overline{X})^2 = 100 \qquad \sum_{i=1}^n (Y_i - \overline{Y})^2 = 200$$
$$\sum_{i=1}^n (X_i - \overline{X})(Y_i - \overline{Y}) = 100 \qquad \text{SSR} = 100$$

(a) (5 marks) Complete the following ANOVA table:

Source	df	SS	MS	F
Regression				
Error				
Total				

(b) (3 marks) Estimate the slope and give a 95% confidence interval for β_1.

(c) (2 marks) Use the ANOVA table to test $H_0 : \beta_1 = 0$ versus $H_1 : \beta_1 \neq 0$.

(d) (3 marks) Give a 90% prediction interval for a new observation at $X = 50$.

2. The data in this question were collected as part of a study of the adult female Dungeness crab. While planning fishing restrictions to control crab populations, biologists want to study the growth rate of crabs. The data are measurements of the widest part of the crabs' shells, in millimeters. Crabs molt regularly, casting off their old shells and growing new ones. Of particular interest is predicting the size of the shell before molting (variable name: `presize`) having observed the size of the shell after the crab molted (variable name: `postsize`).

   SAS output is given below for the regression of `postsize` on `presize` for 342 adult female crabs raised in a laboratory setting.

```
                    The REG Procedure
           Number of Observations Read        342
           Number of Observations Used        342

                  Descriptive Statistics
                              Uncorrected                 Standard
Variable         Sum       Mean          SS     Variance   Deviation
Intercept  342.00000    1.00000   342.00000            0           0
postsize       49151  143.71696     7096984     97.13490     9.85570
presize        44133  129.04357     5736616    121.84434    11.03831

                    The REG Procedure
                     Model: MODEL1
               Dependent Variable: presize

                   Analysis of Variance
                            Sum of          Mean
Source              DF     Squares        Square   F Value   Pr > F
Model                1       40192         40192   10072.0   <.0001
Error              340  1356.76275       3.99048
Corrected Total    341       41549

             Root MSE            1.99762   R-Square    0.9673
             Dependent Mean    129.04357   Adj R-Sq    0.9672
             Coeff Var           1.54802

                   Parameter Estimates
                     Parameter      Standard
     Variable    DF   Estimate         Error   t Value   Pr > |t|
     Intercept    1  -29.26843       1.58114    -18.51     <.0001
     postsize     1    1.10155       0.01098    100.36     <.0001
```

Questions related to these data are on the next three pages.

(a) (7 marks) Complete the chart below.

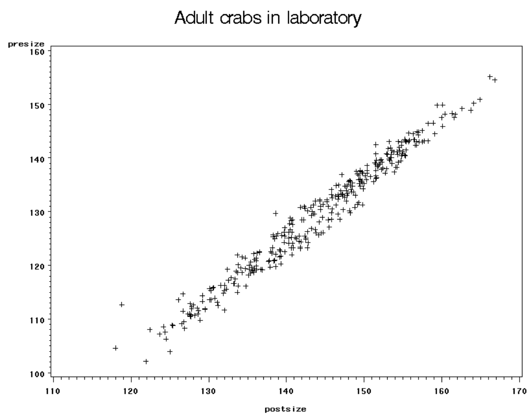
Statistic	Observed Value
Slope of line	
Correlation between `postsize` and `presize`	
Average change in `presize` for an increase of 10 mm in `postsize`	
Estimate of `presize` when `postsize` is 130 mm	
Estimated variance of the intercept	
Test statistic for test of $H_0 : \beta_1 = 0$ versus $H_a : \beta_1 \neq 0$	
Estimate of σ^2	

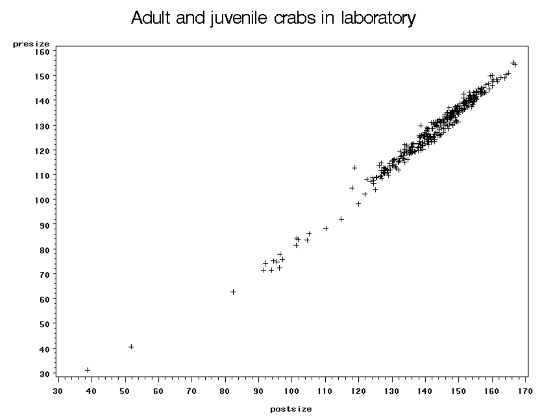
(b) (2 marks) Assume the usual simple linear regression assumptions hold. What is the distribution of the values of `presize` for crabs with `postsize` of 130 mm?

(c) (3 marks) Find the limits of the 95% Working-Hotelling confidence interval when `postsize` is 130 mm.

(d) (2 marks) Explain the meaning of the Working-Hotelling confidence interval in part (c).

(e) (4 marks) Two scatterplots are given below. The first is the plot of `presize` versus `postsize` for the 342 adult female crabs considered in the analysis above. The second plot includes these 342 crabs, as well as an additional 19 juvenile crabs.





How will adding the juvenile crabs to the regression affect the estimated slope and the value of R^2? Explain.

(f) (3 marks) A quadratic model was also fit to the original data for the 342 adult females (new variable: `postsize2` is the square of `postsize`) and some resulting SAS output is given below.

Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	2	40211	20106	5096.25	<.0001
Error	339	1337.42536	3.94521		
Corrected Total	341	41549			

Root MSE	1.98625	R-Square	0.9678
Dependent Mean	129.04357	Adj R-Sq	0.9676
Coeff Var	1.53921		

Parameter Estimates

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	13.40308	19.33811	0.69	0.4887
postsize	1	0.50049	0.27171	1.84	0.0663
postsize2	1	0.00211	0.00095144	2.21	0.0275

Find the coefficient of partial determination for the inclusion of the quadratic term in the model given the linear term, and interpret its meaning.

3. Assume we are fitting the multiple linear regression model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ where $\mathbf{Y}$ and $\boldsymbol{\epsilon}$ are $n \times 1$ vectors, $\mathbf{X}$ is a $n \times (k+1)$ matrix, and $\boldsymbol{\beta}$ is a $(k+1) \times 1$ vector. Recall that the least squares estimate of $\boldsymbol{\beta}$ is $\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$ and $\text{Cov}(\mathbf{b}) = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}$. Assume the Gauss-Markov conditions apply.

   (a) (2 marks) State the Gauss-Markov conditions for this model.

   (b) (2 marks) Show that $\mathbf{b}$ is unbiased for $\boldsymbol{\beta}$.

   (c) (4 marks) The hat matrix is $\mathbf{H} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$. Show that $\mathbf{H}$ is symmetric and idempotent.

   (d) (4 marks) Suppose we express the jth independent variable X_j in centimeters instead of meters (all other variables don't change). (There are 100 centimeters in a meter.) What will happen to b_j and the variance of b_j?

   (e) (3 marks) What would happen to b_j and the variance of b_j if all of the values $X_{1j}, \ldots, X_{nj}$ of the jth independent variable were replaced by numbers that were nearly constant?

4. Data are available for 67 construction crews on the number of lost days of work due to injury over a period of one year. We are interested in understanding whether the number of lost days per person (variable name: `lostdays_pp`, the average number of lost days per person on the crew) is related to the size of the work crew (variable name: `size`, the number of people on the crew) and the experience of the foreman in charge of the crew (variable name: `f_exp`, measured in years).

   Some output from SAS is given below.

```
                    The REG Procedure
                      Model: MODEL1
              Dependent Variable: lostdays_pp

           Number of Observations Read          67
           Number of Observations Used          67

                   Analysis of Variance
                             Sum of           Mean
Source              DF      Squares         Square    F Value    Pr > F
Model                2    215.73625      107.86812      11.37    <.0001
Error               64    607.32882        9.48951
Corrected Total     66    823.06507

             Root MSE              3.08051    R-Square     0.2621
             Dependent Mean        3.54168    Adj R-Sq     0.2391
             Coeff Var            86.97865

                    Parameter Estimates
                        Parameter       Standard
    Variable      DF     Estimate          Error    t Value    Pr > |t|
    Intercept      1      0.20769        1.28855       0.16      0.8725
    size           1      0.71201        0.14981       4.75      <.0001
    f_exp          1     -0.26702        0.08234      -3.24      0.0019
```

   (a) (2 marks) Explain the meaning of the coefficient of `size` for a non-statistician who is putting together construction crews for future jobs.

   (b) (3 marks) What are the null and alternative hypotheses of the analysis of variance F test? What do you conclude? Relate your answer to lost days for construction crews.

(c) (2 marks) In addition to the model in the SAS output above, a model was fit including an additional term which is the product of `size` and `f_exp` (variable name: `fexpsize`). What is the purpose of including this additional term?

(d) (2 marks) Output from SAS for the model in part (c) is given below.

Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	3	241.74285	80.58095	8.73	<.0001
Error	63	581.32221	9.22734		
Corrected Total	66	823.06507			

Root MSE	3.03765	R-Square	0.2937
Dependent Mean	3.54168	Adj R-Sq	0.2601
Coeff Var	85.76871		

Parameter Estimates

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	5.75326	3.53921	1.63	0.1090
size	1	0.14136	0.37063	0.38	0.7042
f_exp	1	-0.62886	0.23032	-2.73	0.0082
fexpsize	1	0.03466	0.02064	1.68	0.0981

Based on this model, does size of construction crew have an effect on the number of lost days per person? Explain.

(e) (2 marks) Which of the two models fit to these data do you prefer? Justify your answer with appropriate statistics.

5. An experimenter wished to compare three different drug products (labelled A, B, and C) for combatting a virus. Four different dosages (0.2, 0.4, 0.8, and 1.0 μg) of each of the drugs were compared. Each of the 12 treatment combinations (3 drug products times 4 dosages) were applied to a culture of the virus and the rates of reduction in the number of cells of the virus were recorded.

   Some output from SAS is below.

```
                    The REG Procedure
                Dependent Variable: rate
          Number of Observations Read          12
          Number of Observations Used          12

                   Analysis of Variance
                              Sum of           Mean
Source                DF     Squares         Square    F Value    Pr > F
Model                  5   129.19267       25.83853      12.00    0.0044
Error                  6    12.91400        2.15233
Corrected Total       11   142.10667

             Root MSE              1.46708    R-Square     0.9091
             Dependent Mean        7.16667    Adj R-Sq     0.8334
             Coeff Var            20.47093

                   Parameter Estimates
                     Parameter      Standard
Variable      DF      Estimate         Error   t Value   Pr > |t|     Type I SS
Intercept      1       2.11000       1.57327      1.34     0.2284     616.33333
dose           1       3.65000       2.31966      1.57     0.1667      61.06133
dose_drugA     1       7.55000       3.28050      2.30     0.0610      44.00014
dose_drugB     1       2.90000       3.28050      0.88     0.4107      23.00000
drugA          1       0.72000       2.22494      0.32     0.7572       0.00419
drugB          1       1.61000       2.22494      0.72     0.4965       1.12700
```

   (a) (4 marks) Write the model that was fit in the SAS output above, defining all variables.

(b) (1 mark) What is the estimated relationship between `rate` and `dose` for drug C?

(c) (2 marks) Explain why it would seem reasonable to assume that the three linear models for the relationships between `rate` and `dose` for the three drugs have a common intercept. Show how to change the regression model from part (a) to reflect this.

(d) (4 marks) Carry out an appropriate statistical test to test the assumption of equal intercepts.

(e) (3 marks) Can you also test whether the relationship between `rate` and `dose` is the same for all three drugs from the given output? If yes, briefly explain how (although you do not need to actually carry out the test). If not, explain what additional information you need.

6. (10 marks, 2 for each part) Sketch an example of a residual plot that would result from a regression for each of the following situations. Indicate what you are plotting on your axes.

   (a) The data has one large influential outlier.

   (b) The data has one large non-influential outlier.

   (c) A log transformation of Y is appropriate.

   (d) The distribution of the residuals is right-skewed.

   (e) A simple linear regression of Y on X was carried out but a model with both a X term and an X^2 term is appropriate.

7. For each of the following questions a short answer is required.

   (a) (2 marks) Is adjusted R^2 always less than R^2? Explain.

   (b) (2 marks) In a simple linear regression, suppose the goal is to get a good estimate of the slope. What is the advantage of increasing the standard deviation of the X's?

   (c) (2 marks) A linear regression is carried out and the plot of Y versus the predicted values is increasing. What should be done? Why?

   (d) (3 marks) Explain the difference between a prediction interval and a calibration interval and how you would decide which to use.

   (e) (2 marks) In examining a normal quantile plot of residuals, why are the values on the extreme left and right often of more interest than the values in the centre?

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Simple regression formulae

$$b_1 = \frac{\sum(X_i-\overline{X})(Y_i-\overline{Y})}{\sum(X_i-\overline{X})^2} = \frac{\sum X_iY_i - n\overline{XY}}{\sum(X_i-\overline{X})^2}$$

$$b_0 = \overline{Y} - b_1\overline{X}$$

$$\text{Var}(b_1) = \frac{\sigma^2}{\sum(X_i-\overline{X})^2}$$

$$\text{Var}(b_0) = \sigma^2\left(\frac{1}{n} + \frac{\overline{X}^2}{\sum(X_i-\overline{X})^2}\right)$$

$$\text{Cov}(b_0, b_1) = -\frac{\sigma^2\overline{X}}{\sum(X_i-\overline{X})^2}$$

$$\text{SSTO} = \sum(Y_i - \overline{Y})^2$$

$$\text{SSE} = \sum(Y_i - \hat{Y}_i)^2$$

$$\text{SSR} = b_1^2\sum(X_i - \overline{X})^2 = \sum(\hat{Y}_i - \overline{Y})^2$$

$$\sigma^2\{\hat{Y}_h\} = \text{Var}(\hat{Y}_h) = \sigma^2\left(\frac{1}{n} + \frac{(X_h-\overline{X})^2}{\sum(X_i-\overline{X})^2}\right)$$

$$\sigma^2\{\text{pred}\} = \text{Var}(Y_h - \hat{Y}_h) = \sigma^2\left(1 + \frac{1}{n} + \frac{(X_h-\overline{X})^2}{\sum(X_i-\overline{X})^2}\right)$$

$$\hat{X}_h \pm \frac{t_{n-2,1-\alpha/2}}{|b_1|} * \text{appropriate s.e.}$$

(valid approximation if $\frac{t^2s^2}{b_1^2\sum(X_i-\overline{X})^2}$ is small)

Working-Hotelling coefficient:

$$W = \sqrt{2F_{2,n-2;1-\alpha}}$$

$$r = \frac{\sum(X_i-\overline{X})(Y_i-\overline{Y})}{\sqrt{\sum(X_i-\overline{X})^2\sum(Y_i-\overline{Y})^2}}$$

Regression in matrix terms

$$\text{Cov}(\mathbf{X}) = \text{E}[(\mathbf{X} - \text{E}\mathbf{X})(\mathbf{X} - \text{E}\mathbf{X})'] = \text{E}(\mathbf{X}\mathbf{X}') - (\text{E}\mathbf{X})(\text{E}\mathbf{X})'$$

$$\text{Cov}(\mathbf{AX}) = \mathbf{A}\text{Cov}(\mathbf{X})\mathbf{A}'$$

$$\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$$

$$\text{Cov}(\mathbf{b}) = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}$$

$$\hat{\mathbf{Y}} = \mathbf{Xb} = \mathbf{HY}$$

$$\mathbf{e} = \mathbf{Y} - \hat{\mathbf{Y}} = (\mathbf{I} - \mathbf{H})\mathbf{Y}$$

$$\mathbf{H} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$$

$$\text{SSR} = \mathbf{Y}'(\mathbf{H} - \tfrac{1}{n}\mathbf{J})\mathbf{Y}$$

$$\text{SSE} = \mathbf{Y}'(\mathbf{I} - \mathbf{H})\mathbf{Y}$$

$$\text{SSTO} = \mathbf{Y}'(\mathbf{I} - \tfrac{1}{n}\mathbf{J})\mathbf{Y}$$

$$\sigma^2\{\hat{Y}_h\} = \text{Var}(\hat{Y}_h) = \sigma^2\mathbf{X}_h'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}_h$$

$$\sigma^2\{\text{pred}\} = \text{Var}(Y_h - \hat{Y}_h) = \sigma^2(1 + \mathbf{X}_h'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}_h)$$

$$R^2_{\text{adj}} = 1 - (n-1)\frac{MSE}{SSTO}$$

$$C_p = \frac{SSE_p}{MSE_P} - (n - 2p)$$

$$PRESS_p = \sum(Y_i - \hat{Y}_{i(i)})^2$$

Total pages 17
Total marks 95